PHILOSOPHICAL TRANSACTIONS OF THE ROYAL SOCIETY A
MATHEMATICAL, PHYSICAL & ENGINEERING SCIENCES

Relaxation Methods Applied to Engineering Problems. VIID. Stress Distributions in Elastic Solids of Revolution

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Phil. Trans. R. Soc. Lond. A 1945 **239**, 501-537
doi: 10.1098/rsta.1945.0004

RELAXATION METHODS APPLIED TO ENGINEERING PROBLEMS

VII D. STRESS DISTRIBUTIONS IN ELASTIC SOLIDS OF REVOLUTION

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(*Received* 23 *December* 1942)

Relaxation methods are employed to solve, *without restriction on the form of the generating curve*, the following problems relating to solids of revolution: (1) torsional stresses in an incomplete tore, (2) torsional stresses in a circular shaft of non-uniform diameter, (3) axially symmetrical stresses in a complete solid of revolution, (4) flexural stresses in an incomplete tore, (5) shearing and flexural stresses in a toroidal 'hook'. Accuracy sufficient for all practical purposes is attained in every case.

Introduction

1. A recent paper (Southwell 1942) obtained directly (i.e. without explicit reference to strains) formal solutions of several problems concerned with solids of revolution, viz.†

Problem (i, 1) *a*: Torsional stresses in an incomplete tore.

Problem (i, 1) *b*: Torsional stresses in a circular shaft having non-uniform diameter.

Problem (i, 2) *a*: Axially symmetrical strain in a complete solid of revolution.

Problem (i, 2) *b*: Flexural stresses in an incomplete tore.

Problem (ii): Shearing and flexural stresses in a toroidal 'hook'.

Each formal solution consisted in a formulation of boundary conditions, and of expressions for the stress components, in terms of one or more 'stress functions' governed by partial differential equations in r and z. Oz is assumed to coincide with the axis of symmetry of the solid, whose cross-section is defined by a specified curve, or curves, in the z-r plane (figure 1). Evaluation of the stress function (or functions) is a problem special to each particular boundary.

This paper extends the range of Relaxation Methods to the computational problems thus presented. Like previous papers in the series it starts by substituting approximate (finite-difference) relations for the exact (differential) equations; but the subsequent treatment is novel in that different 'relaxation patterns' are employed (to liquidate 'residual forces') at different points of the relaxation net.

For brevity the discussion presumes a knowledge of processes described in earlier papers of this series, viz. Part III (Christopherson & Southwell 1938), Part V (Gandy &

† The numbering is that used in the paper cited. There it was stated that formal solutions had been given previously to all of these problems except (ii), but that the new solutions had been found better suited to computation by Relaxation Methods.

[Issued 31 December 1943—Published 10 October 1945

Southwell 1940), Part VII (Shaw & Southwell 1941), Part VIIA (Fox & Southwell 1945), Part VIIB (Christopherson, Fox, Green, Shaw & Southwell 1945), Part VIIC (Allen, Fox, Motz & Southwell 1945).

Section I. Formal solutions

2. In this section the formal solutions are presented with appropriate references to the earlier paper, asterisks being used to distinguish equation nos., etc., in Southwell (1942).

Problem (i, 1) *a: Torsional stresses in an incomplete tore* (§§ 9–10*)

3. The stress components $\widehat{rr}$, $\widehat{\theta\theta}$, $\widehat{zz}$, $\widehat{zr}$ vanish everywhere. The other two have the expressions

$$\widehat{r\theta} = \frac{A}{r^2}\frac{\partial\phi}{\partial z}, \qquad \widehat{\theta z} = -\frac{A}{r^2}\frac{\partial\phi}{\partial r} \tag{1}$$

in terms of a function ϕ which is governed by

$$\left[\frac{\partial^2}{\partial r^2} - \frac{3}{r}\frac{\partial}{\partial r} + \frac{\partial^2}{\partial z^2}\right]\phi = 1 \qquad (27\,a)^* \tag{2}$$

at every point in an axial cross-section of the tore, and by

$$\phi = 0 \qquad (31)^* \tag{3}$$

at every point on its boundary. The resultant action for the whole cross-section of $\widehat{r\theta}$, $\widehat{\theta z}$ as given by (1) is a force $\mathbf{Z}$ directed along the axis of revolution Oz and given by

$$\mathbf{Z} = \tfrac{1}{2}A\oint\left\{\cos(r,\nu)\log r - \frac{1}{r}\frac{\partial\phi}{\partial\nu}\right\}ds, \qquad (32)^* \tag{4}$$

ν denoting the normal to the cross-section drawn outwards. Figure 1 explains the senses of ν and s in relation to z and r.

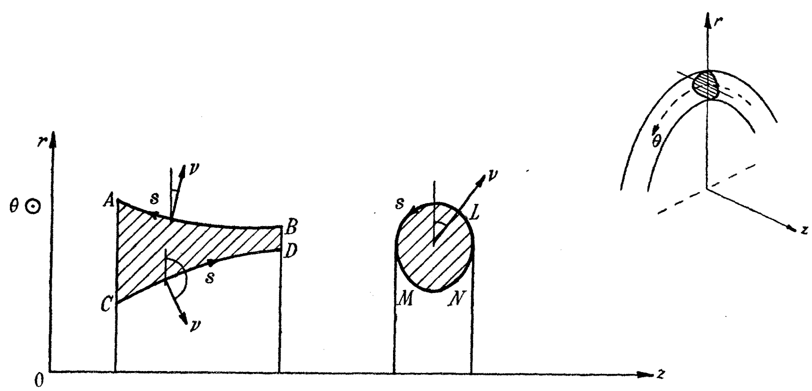

Figure 1

This solution accordingly gives (§ 10*) "...a stress system which would result from shearing actions, suitably distributed over the terminal sections, of which the resultants are directed along the axis of the tore. Actions of this kind are exerted when a helical spring of small pitch withstands tension or compression...".

Problem (i, 1) *b: Torsional stresses in a circular shaft having non-uniform diameter* (§ 11*)

4. The two non-zero stress components are given by (1) as before, but ϕ is now governed by

$$\left[\frac{\partial^2}{\partial r^2}-\frac{3}{r}\frac{\partial}{\partial r}+\frac{\partial^2}{\partial z^2}\right]\phi = 0, \qquad (33)^* \tag{5}$$

so would vanish everywhere if (all boundary tractions being zero) the boundary condition were (3) as before. Here, of course, the stress system results from tractions applied to some part (at least) of the boundary surface, and in consequence (3) is replaced by

$$A\frac{\partial\phi}{\partial s} = -r^2 . \widehat{\nu\theta} = -r^2\{\widehat{r\theta}\cos(r,\nu)+\widehat{\theta z}\sin(r,\nu)\}, \tag{6}$$

in which $\widehat{\nu\theta}$ is specified. Usually $\widehat{\nu\theta}$ will be zero except at two 'cross-sections' perpendicular to Oz the axis of revolution (figure 1), where (6) reduces to

$$\left(\pm A\frac{\partial\phi}{\partial s}=\right) A\frac{\partial\phi}{\partial r} = -r^2 . \widehat{\theta z}, \tag{7}$$

$\widehat{\theta z}$ being specified.

Problem (i, 2) *a: Axially symmetrical strain in a complete solid of revolution* (§§ 16–17*)

5. The stress components $\widehat{r\theta}$, $\widehat{\theta z}$ vanish everywhere, and the other four are expressed as under:

$$\left.\begin{aligned}
\widehat{rr}+\widehat{\theta\theta} &= \frac{1}{r}\frac{\partial^3\lambda}{\partial r\,\partial z^2},\\
\widehat{rr}-\widehat{\theta\theta} &= r\frac{\partial}{\partial r}\frac{1}{r^2}\left(\frac{\partial^2\lambda}{\partial z^2}-2\sigma\vartheta^2\lambda\right),\\
\widehat{zz} &= -\frac{1}{r}\frac{\partial}{\partial r}\left\{\frac{\partial^2\lambda}{\partial z^2}-(1+\sigma)\,\vartheta^2\lambda\right\},\\
\widehat{zr} &= \frac{1}{r}\frac{\partial}{\partial z}\left\{\frac{\partial^2\lambda}{\partial z^2}-(1+\sigma)\,\vartheta^2\lambda\right\},\\
\Theta &= \frac{1+\sigma}{r}\frac{\partial}{\partial r}\vartheta^2\lambda,
\end{aligned}\right\}\qquad (52)^* \tag{8}$$

λ denoting a function of z and r which satisfies the equation

$$\vartheta^4\lambda = 0 \qquad (45)^*$$

when ϑ^2 stands for the operator

$$\left[\frac{\partial^2}{\partial r^2}-\frac{1}{r}\frac{\partial}{\partial r}+\frac{\partial^2}{\partial z^2}\right]. \qquad (38)^* \tag{9}$$

Writing $$\chi = \vartheta^2\lambda, \qquad \psi = \frac{\partial^2\lambda}{\partial z^2}-(1+\sigma)\,\vartheta^2\lambda, \qquad (54)^* \tag{10}$$

we may state the boundary conditions (53)* as

$$\widehat{\nu r} = \frac{\partial}{\partial \nu}\left(\frac{\psi}{r}\right) + \cos(r,\nu)\left[\frac{\partial}{\partial r} + \frac{\sigma}{r}\right]\frac{\chi}{r}, \quad -\widehat{\nu z} = \frac{1}{r}\frac{\partial \psi}{\partial s}, \tag{11}$$

$\widehat{\nu r}$, $\widehat{\nu z}$ being specified. The second of these can be integrated to impose boundary values on ψ (the constant of integration is without importance, and it can be shown that ψ is acyclic). If ψ can be determined, then the first of (11) imposes a single boundary condition on χ, which satisfies the equation

$$\vartheta^2 \chi = 0$$

in virtue of the first of (9). We observe that

$$\vartheta^2 \psi = \frac{\partial^2 \chi}{\partial z^2} \tag{12}$$

according to (10).

Extension of the solution to include allowance for accelerations

6. The foregoing solution presumes that no body forces are operative. But important problems are presented, satisfying the condition of axial symmetry, in which 'reversed effective' body forces arise (on d'Alembert's principle) either (*a*) because the body undergoes uniform acceleration f_z in the direction Oz, or (*b*) because the body rotates about Oz with uniform angular velocity ω and in consequence any particle distant r from that axis is subjected to a radial acceleration $f_r = -\omega^2 r$. Both cases can be treated as statical problems if body-force terms

$$\rho F_r = \rho\omega^2 r, \qquad \rho F_z = -\rho f_z \quad (f_z \text{ const.}) \tag{i}$$

are added to the first and third of (13)*, the equations of equilibrium for Problem (i, 2) *a*. These body forces can be derived according to the formulae

$$F_r = -\frac{\partial \Omega}{\partial r}, \quad F_z = -\frac{\partial \Omega}{\partial z}$$

from a potential function

$$\Omega = z f_z - \tfrac{1}{2}\omega^2 r^2 + \text{const.} \tag{ii}$$

(cf. Southwell 1941, § 436).

7. The conditions of compatibility—namely, equations (10)*—are altered by the occurrence of the body forces. Treatment on the lines of Love (1927), § 92, shows that when these are derivable from a potential function Ω, then the three equations typified by

$$\nabla^2 X_x + \frac{1}{1+\sigma}\frac{\partial^2 \Theta}{\partial x^2} = 0 \tag{2)*$$

are replaced by equations of which

$$\nabla^2 X_x + \frac{1}{1+\sigma}\frac{\partial^2 \Theta}{\partial x^2} = 2\rho\frac{\partial^2 \Omega}{\partial x^2} + \frac{\sigma}{1-\sigma}\rho\nabla^2\Omega \tag{iii}$$

is typical, and the three equations typified by

$$\nabla^2 Y_z + \frac{1}{1+\sigma}\frac{\partial^2 \Theta}{\partial y\,\partial z} = 0 \qquad (3)^*$$

are replaced by equations of which

$$\nabla^2 Y_z + \frac{1}{1+\sigma}\frac{\partial^2 \Theta}{\partial y\,\partial z} = 2\rho\frac{\partial^2 \Omega}{\partial y\,\partial z} \qquad \text{(iv)}$$

is typical.

Here, according to (ii),

$$\dot{\Omega} = z f_z - \tfrac{1}{2}\omega^2(x^2+y^2) + \text{const.},$$

so $\partial^2\Omega/\partial y\,\partial z = 0$ and the three equations of type (3)* are unaltered; but

$$\frac{\partial^2\Omega}{\partial x^2} = \frac{\partial^2\Omega}{\partial y^2} = -\omega^2, \quad \frac{\partial^2\Omega}{\partial z^2} = 0, \quad \text{therefore} \quad \nabla^2\Omega = -2\omega^2, \qquad \text{(v)}$$

so the three equations of type (2)* are altered to

$$\left.\begin{aligned} \nabla^2 X_x + \frac{1}{1+\sigma}\frac{\partial^2\Theta}{\partial x^2} + \frac{2\rho\omega^2}{1-\sigma} &= 0,\\ \nabla^2 Y_y + \frac{1}{1+\sigma}\frac{\partial^2\Theta}{\partial y^2} + \frac{2\rho\omega^2}{1-\sigma} &= 0,\\ \nabla^2 Z_z + \frac{1}{1+\sigma}\frac{\partial^2\Theta}{\partial z^2} + \frac{2\sigma\rho\omega^2}{1-\sigma} &= 0. \end{aligned}\right\} \qquad \text{(vi)}$$

When these are transformed to cylindrical co-ordinates in the manner of Southwell (1942), §§ 1–2, the first and fourth of the resulting equations (14)* are altered to

$$\nabla^2(\widehat{rr}+\widehat{\theta\theta}) + \left[\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r}\right]\left(\frac{\Theta}{1+\sigma}\right) + \frac{4\rho\omega^2}{1-\sigma} = 0$$

and to

$$\nabla^2\widehat{zz} + \frac{\partial^2}{\partial z^2}\left(\frac{\Theta}{1+\sigma}\right) + \frac{2\sigma\rho\omega^2}{1-\sigma} = 0, \qquad (14)^*\,\text{A}$$

the other four are unaltered.

8. Treating the density ρ as uniform, we shall satisfy the amended conditions of equilibrium if we add to $\widehat{rr}$, $\widehat{\theta\theta}$, $\widehat{zz}$ as given by (8) increments $\Delta\widehat{rr}$, $\Delta\widehat{\theta\theta}$, $\Delta\widehat{zz}$ given by

$$\Delta\widehat{rr} = -\frac{1}{8}\frac{3-2\sigma}{1-\sigma}\rho\omega^2 r^2, \quad \Delta\widehat{\theta\theta} = -\frac{1}{8}\frac{1+2\sigma}{1-\sigma}\rho\omega^2 r^2, \quad \Delta\widehat{zz} = -\frac{1}{2}\frac{\sigma}{1-\sigma}\rho\omega^2 r^2 + \rho f_z z,$$

thereby adding to Θ an increment

$$\Delta\Theta = -\frac{1}{2}\frac{1+\sigma}{1-\sigma}\rho\omega^2 r^2 + \rho f_z z. \qquad \text{(vii)}$$

We shall, moreover, satisfy both of (14)* A, together with the other (unamended) equations (14)*; i.e. we shall satisfy *all* the conditions of our problem. Accordingly we now combine (vii) with (8) of § 5 in the expressions

$$\left.\begin{aligned}
\widehat{rr}+\widehat{\theta\theta} &= \frac{1}{r}\frac{\partial^3\lambda}{\partial r\,\partial z^2}-\frac{1}{2}\frac{\rho\omega^2r^2}{1-\sigma},\\
\widehat{rr}-\widehat{\theta\theta} &= r\frac{\partial}{\partial r}\frac{1}{r^2}\left(\frac{\partial^2\lambda}{\partial z^2}-2\sigma\vartheta^2\lambda\right)-\frac{1}{4}\frac{1-2\sigma}{1-\sigma}\rho\omega^2r^2,\\
\widehat{zz} &= -\frac{1}{r}\frac{\partial}{\partial r}\left\{\frac{\partial^2\lambda}{\partial z^2}-(1+\sigma)\,\vartheta^2\lambda\right\}-\frac{1}{2}\frac{\sigma}{1-\sigma}\rho\omega^2r^2+\rho f_z z,\\
\widehat{zr} &= \frac{1}{r}\frac{\partial}{\partial z}\left\{\frac{\partial^2\lambda}{\partial z^2}-(1+\sigma)\,\vartheta^2\lambda\right\},\\
\Theta &= \frac{1+\sigma}{r}\frac{\partial}{\partial r}\vartheta^2\lambda-\frac{1}{2}\frac{1+\sigma}{1-\sigma}\rho\omega^2r^2+\rho f_z z,
\end{aligned}\right\} \qquad (8)\,\mathrm{A}$$

which apply to both of the problems described in § 6.

The stress function λ satisfies (9) as before, so (10) and (12), § 5, still hold; but now the boundary conditions (11) are replaced by

$$\left.\begin{aligned}
\widehat{\nu r}+\frac{1}{8}\frac{3-2\sigma}{1-\sigma}\rho\omega^2r^2\cos(r,\nu) &= \frac{\partial}{\partial\nu}\left(\frac{\psi}{r}\right)+\cos(r,\nu)\left[\frac{\partial}{\partial r}+\frac{\sigma}{r}\right]\frac{\chi}{r},\\
-\widehat{\nu z}+\left(\rho f_z z-\frac{1}{2}\frac{\sigma}{1-\sigma}\rho\omega^2r^2\right)\sin(r,\nu) &= \frac{1}{r}\frac{\partial\psi}{\partial s}.
\end{aligned}\right\} \qquad (11\,\mathrm{A})$$

Problem (i, 2) *b: Flexural stresses in an incomplete tore* (§§ 18–19*)

9. The formal solution of this problem (in which no allowance has to be made for accelerations) is similar to that of § 5. $\widehat{r\theta}$ and $\widehat{\theta z}$ vanish everywhere, and the other stress components are given by

$$\left.\begin{aligned}
\widehat{rr}+\widehat{\theta\theta} &= \frac{1}{r}\frac{\partial^3\mu}{\partial r\,\partial z^2},\\
\widehat{rr}-\widehat{\theta\theta} &= r\frac{\partial}{\partial r}\frac{1}{r^2}\left(\frac{\partial^2\mu}{\partial z^2}-2\sigma\vartheta^2\mu\right)+2B\left(\sigma+\frac{z^2}{r^2}\right),\\
\widehat{zz} &= \frac{1}{r}\frac{\partial}{\partial r}\left\{-\frac{\partial^2\mu}{\partial z^2}+(1+\sigma)\,\vartheta^2\mu\right\},\\
\widehat{zr} &= -\frac{1}{r}\frac{\partial}{\partial z}\left\{-\frac{\partial^2\mu}{\partial z^2}+(1+\sigma)\,\vartheta^2\mu\right\},\\
\Theta &= \frac{1+\sigma}{r}\frac{\partial}{\partial r}\vartheta^2\mu,
\end{aligned}\right\} \qquad (56)^* \qquad (13)$$

μ denoting a function of z and r which satisfies the equation

$$\vartheta^4\mu = 2B \qquad (57)^* \tag{14}$$

when ϑ^2 has the same significance as in (9).

Writing $\chi = \vartheta^2\mu, \quad \psi = \dfrac{\partial^2\mu}{\partial z^2} - (1+\sigma)\,\vartheta^2\mu,$
so that $\vartheta^2\chi = 2B,$ (15)

we can state the boundary conditions (58)* as

$$\left.\begin{aligned} \psi &= 0,\\ \frac{\partial}{\partial\nu}\left(\frac{\psi}{r}\right) + \cos(r,\nu)\left\{\left[\frac{\partial}{\partial r}+\frac{\sigma}{r}\right]\left(\frac{\chi}{r}\right) + B\left(\sigma+\frac{z^2}{r^2}\right)\right\} &= 0, \end{aligned}\right\} \tag{16}$$

the constant of integration in the first condition being without importance (cf. § 5).

Problem (ii): *Shearing and flexural stresses in a toroidal 'hook'* (§§ 24–7*)

10. This problem does not appear to have been treated previously, except in particular cases (e.g. as a problem in plane stress). The six stress components are given by†

$$\left.\begin{aligned} &r^2.\widehat{zz} = \frac{\partial\phi}{\partial r}, \quad r^2.\widehat{zr} = \frac{\partial}{\partial z}\left(\frac{1}{r}\frac{\partial\psi}{\partial r}-\phi\right), \quad r^2.\widehat{\theta z} = \frac{\partial}{\partial z}\left(\left[\frac{\partial^2}{\partial r^2}-\frac{2}{r}\frac{\partial}{\partial r}\right]\psi+\phi\right), \qquad (61)^*\\ &\left.\begin{aligned} r^3.\widehat{rr} &= Q(r^2+\sigma z^2) - \sigma A(zr^2+\tfrac{1}{3}z^3) - \left[r\frac{\partial}{\partial r}-3\right]\phi\\ &\quad -2\frac{\partial^2\psi}{\partial z^2} - \left[r\frac{\partial}{\partial r}-(3-\sigma)\right]\vartheta_3^2\psi,\\ r^3.\widehat{\theta\theta} &= -Q(r^2+\sigma z^2) - A\{(2+\sigma)\,zr^2 - \tfrac{1}{3}\sigma z^3\} - 3\phi\\ &\quad +2\frac{\partial^2\psi}{\partial z^2} - \left[\sigma r\frac{\partial}{\partial r}+(3-\sigma)\right]\vartheta_3^2\psi,\\ -r^3.\widehat{r\theta} &= Q(r^2+\sigma z^2) - \sigma A(zr^2+\tfrac{1}{3}z^3) - \left[r\frac{\partial}{\partial r}-3\right]\phi\\ &\quad +\left[r\frac{\partial}{\partial r}-2\right]\frac{\partial^2\psi}{\partial z^2} - \left[r\frac{\partial}{\partial r}-(3-\sigma)\right]\vartheta_3^2\psi, \end{aligned}\right\}\ (75)^* \end{aligned}\right\} \tag{17}$$

where ϕ and ψ are functions governed by the equations

$$\left.\begin{aligned} &\vartheta_3^2\phi = \frac{\partial^2}{\partial z^2}\vartheta_3^2\psi, \quad \vartheta_3^4\psi = 2(Q+Az) \qquad (65)^*\\ &\left(\vartheta_3^2 \text{ denoting the operator } \left[\frac{\partial^2}{\partial r^2}-\frac{3}{r}\frac{\partial}{\partial r}+\frac{\partial^2}{\partial z^2}\right]\right) \qquad (59)^* \end{aligned}\right\} \tag{18}$$

† The manner of their variation with θ is explained in § 34.

and which satisfy the boundary conditions

$$\left.\begin{aligned}\phi=\frac{\partial^2\psi}{\partial z\,\partial r}=0,\\ \cos(r,\nu)\left\{Q(r^2+\sigma z^2)-\sigma A(zr^2+\tfrac{1}{3}z^3)-2\frac{\partial^2\psi}{\partial z^2}-\left[r\frac{\partial}{\partial r}-(3-\sigma)\right]\vartheta_3^2\psi\right\}=r\frac{\partial\phi}{\partial\nu}.\end{aligned}\right\} \quad (76)^* \quad (19)$$

The constants Q and A are to be determined from the condition that the applied load is

$$\mathbf{W}=-\iint\widehat{r\theta}\,dr\,dz, \qquad (77)^*$$

and has a moment about the origin given by

$$\mathbf{N}=\iint(r\,.\,\widehat{\theta z}-z\,.\,\widehat{r\theta})\,dr\,dz.$$

We observe that in virtue of (18)

$$\vartheta_3^4\phi=0 \quad \text{and} \quad \vartheta_3^2\Psi=-2(Q-Az) \qquad (20)$$

when Ψ stands for the quantity within curly brackets in the last of (19); also that when the boundary is symmetrical about the axis Or (as will be the fact in most problems having practical interest) solutions in which both of ϕ and ψ are respectively even and odd in z correspond with the independent constants Q and A. Then (for symmetrical loading) the Q-solution contributes to **W** but not to **N**, the A-solution to **N** but not to **W.**

Section II. Numerical solutions by the relaxation method

11. We now seek, for some particular shapes of boundary, numerical solutions of the problems formally solved in §§3–10. *Except in one instance* (§23) *we shall assume throughout that σ has the value* 0·3. Our technique will be that developed in earlier papers of

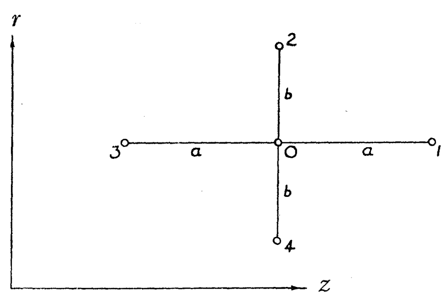

Figure 2

this series: that is to say, (1) for differentials, in the governing equations and in the boundary conditions, we substitute finite-difference approximations as given below; (2) the governing equations thus modified we use to define 'residual forces' at nodal points of a chosen net, and to compute initial values of those forces, also (3) to construct 'relaxation patterns' whereby the residual forces may be 'liquidated' (i.e. reduced to negligible quantities); (4) from the resulting solution, again replacing differentials by their finite-difference approximations, we deduce values of the stress components; (5) by usual methods (cross-plotting, etc.) we construct contour diagrams as graphical representations of our results.

We shall require the following finite-difference approximations, as relating to the typical point 0 of the net shown in figure 2:

$$\left.\begin{aligned} 2a\left(\frac{\partial\psi}{\partial z}\right)_0 &\doteqdot \psi_1-\psi_3, \quad 2b\left(\frac{\partial\psi}{\partial r}\right)_0 \doteqdot \psi_2-\psi_4,\\ a^2\left(\frac{\partial^2\psi}{\partial z^2}\right)_0 &\doteqdot \psi_1+\psi_3-2\psi_0, \quad b^2\left(\frac{\partial^2\psi}{\partial r^2}\right)_0 \doteqdot \psi_2+\psi_4-2\psi_0.\end{aligned}\right\} \tag{21}$$

These expressions provide for the use of a net of rectangular (not necessarily square) mesh, the mesh-side being a in the direction of z, b in the direction of r.

Problem (i, 1) *a: Torsional stresses in an incomplete tore*

12. The wanted function ϕ is governed by equation (2) and is subject to the boundary condition (3) of § 3. Both of these retain their forms when we substitute

$$r' \text{ for } r/L, \quad z' \text{ for } z/L, \quad A' \text{ for } A/L, \quad \phi' \text{ for } \phi/L^2, \tag{22}$$

L denoting some representative dimension of the cross-section; and the same is true of (1), by which the stress components are expressed in terms of ϕ, also of (4)—when ν and s are similarly interpreted—except that $\mathbf{Z}$, on the left-hand side, is now replaced by $\mathbf{Z}/L^2$. This means that every quantity in (1)–(4) may be regarded as 'non-dimensional' (i.e. as purely numerical), provided that (4) is now replaced by

$$2\mathbf{Z}/L^2 = A\oint\left\{\cos(r,\nu)\log r-\frac{1}{r}\frac{\partial\phi}{\partial\nu}\right\}ds, \tag{4) A}$$

so that A, in (1) and (4) A, becomes a multiple of $\mathbf{Z}/L^2$ (a stress); the expressions (1) then define $\widehat{r\theta}$, $\widehat{\theta z}$ as multiples of $\mathbf{Z}/L^2$. *We shall adopt this non-dimensional interpretation of* r, z, ν, A, ϕ, *but without actually inserting dashes in the equations.*

13. Replacing differentials by their finite-difference approximations in accordance with (21), we now replace (2) by

$$\mathbf{F}_0 = \Sigma_{a,4}(\phi)-4\phi_0-\frac{3}{2}\frac{a}{r}(\phi_2-\phi_4)-a^2 = 0, \tag{23}$$

—a relation between the values of ϕ at a typical point 0 and at the four surrounding points 1, 2, 3, 4 of a square-mesh net. The mesh-side a, like r, is now to be regarded as a multiple of the representative dimension, L, defining the chosen net.

Starting with ϕ zero everywhere, we have

$$\mathbf{F} = -a^2, \quad \text{everywhere},$$

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according to (23). Since r/a appears in that equation, the 'relaxation pattern' is different for every row of nodes parallel to Oz. For example, when $r_0 = 3a$ so that $r_1 = r_3 = 3a$, $r_2 = 4a$, $r_4 = 2a$, the effects of unit displacement at 0 (i.e. $\Delta\phi_0 = 1$) are easily seen to be

$$\Delta \mathbf{F}_0 = -4, \quad \Delta \mathbf{F}_1 = \Delta \mathbf{F}_3 = 1, \quad \Delta \mathbf{F}_2 = \tfrac{11}{8}, \quad \Delta \mathbf{F}_4 = \tfrac{1}{4}, \tag{24}$$

—summing nearly *but not exactly*† to zero. Allowance for one or more short strings in an 'irregular star' is easily made (if required)‡ on the basis of 'fictitious nodes' lying outside the boundary.

14. Except that a variety of 'relaxation patterns' is entailed, the liquidation process follows exactly on the lines described in previous papers of this series, and it presented no difficulty in the example which follows. (This was treated by D. N. de G. A.)

Example 1. *Torsion of a tore of square section*

The side of the square section was taken as the governing dimension L, and a was given the values 1/8, 1/16 in successive nets. Figure 3 records the accepted values of ϕ as computed on the final net, also contours of ϕ obtained from cross-plottings.§ It follows from (1) that these contours coincide at every point with the direction of the resultant shear stress: their spacing, on the other hand, is not a direct measure of the intensity of that resultant, but of its intensity multiplied by r^2. Accordingly figure 4 is appended to show the variation of the shear intensity over the cross-section. Figures 3 and 4 in combination present the whole solution.

15. Thom & Orr (1931) attacked the similar problem of a circular shaft of non-uniform diameter (§4) by a process in which (*a*) the governing equation (5) was replaced by its finite-difference approximation, (*b*) this substituted equation was solved *by an iterative process* after (5) had (in effect) been replaced by

$$\left[\frac{\partial^2}{\partial r^2}+\frac{\partial^2}{\partial z^2}\right]\phi_n = \frac{3}{r}\frac{\partial}{\partial r}\phi_{n-1}, \tag{5 A}$$

in which $\phi_1, \phi_2, \ldots, \phi_{n-1}, \phi_n, \ldots$, etc., stand for successive approximations to the wanted function ϕ. A similar attack on Problem (i, 1) *a* in relation to an elliptical section was made some two years ago (by L. F.): the method proved successful, but was slower than the direct method here proposed (§§ 12–14).

Problem (i, 1) *b: Torsional stresses in a circular shaft having non-uniform diameter*

16. In this problem the wanted function ϕ is governed by equation (5) and is subject to the boundary condition (6) of §4. The 'non-dimensional' substitutions (22) make no difference to the form of (5), and since boundary tractions are specified (not a

† This is a new feature of the present paper, due to the occurrence of r in (23).

‡ Irregular stars are not presented in Example 1.

§ In figure 3, ϕ has been multiplied by $-256{,}000$ in order to eliminate negative signs and decimals from the recorded values.

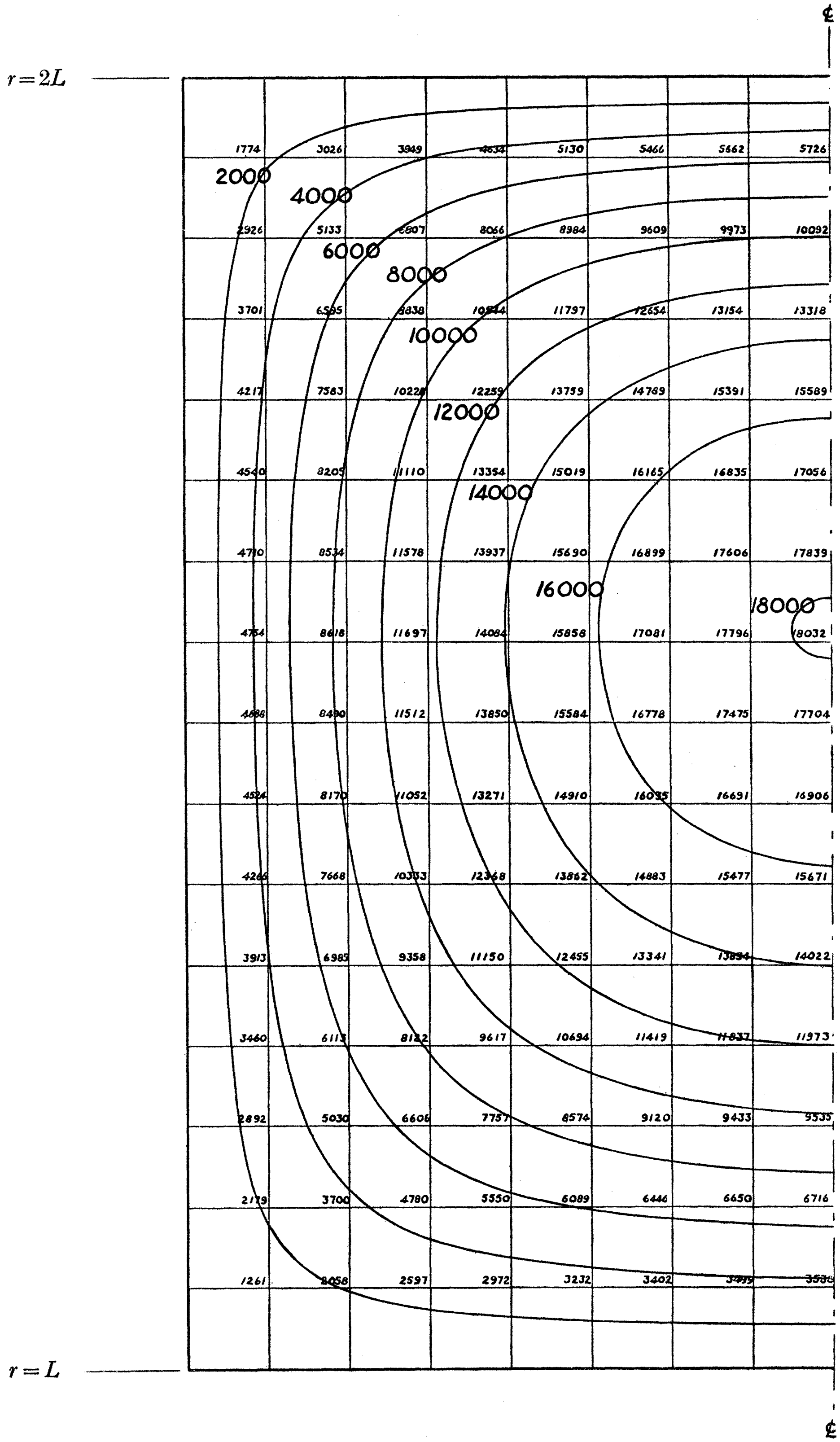
$r = 2L$
$r = L$
2000
4000
6000
8000
10000
12000
14000
16000
18000
1774 3026 3949 4634 5130 5466 5662 5726
2926 5133 6807 8066 8984 9609 9973 10092
3701 6595 8838 10574 11797 12654 13154 13318
4217 7583 10228 12259 13759 14789 15391 15589
4540 8205 11110 13354 15019 16165 16835 17056
4710 8534 11578 13937 15690 16899 17606 17839
4754 8618 11697 14084 15858 17081 17796 18032
4688 8490 11512 13850 15584 16778 17475 17704
4524 8170 11052 13271 14910 16035 16691 16906
4266 7668 10333 12368 13862 14883 15477 15671
3913 6985 9358 11150 12455 13341 13894 14022
3460 6113 8122 9617 10694 11419 11837 11973
2892 5030 6606 7757 8574 9120 9433 9535
2179 3700 4780 5550 6089 6446 6650 6716
1261 2058 2597 2972 3232 3402 3499 3538

FIGURE 3

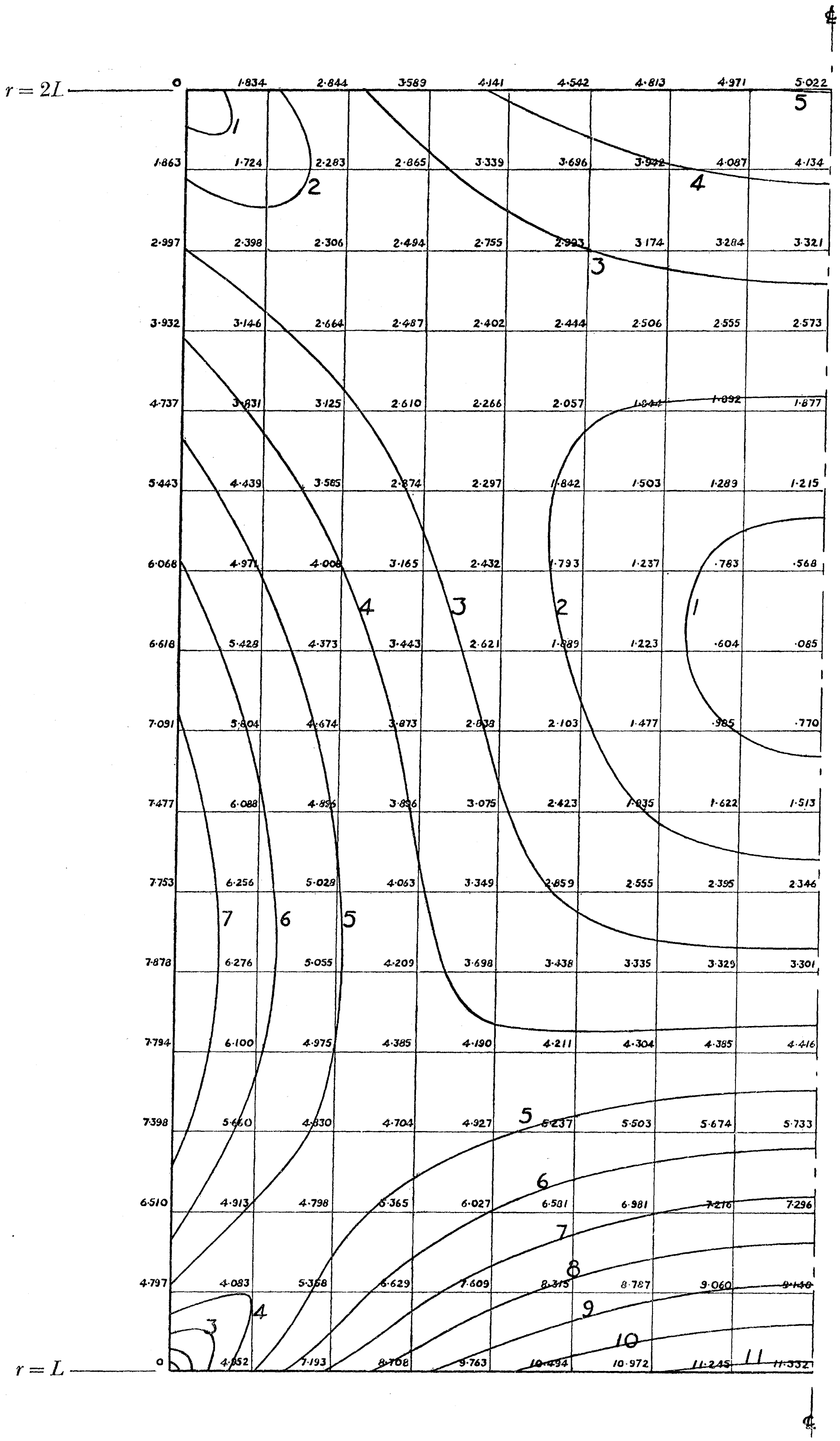

Figure 4

resultant action as in Problem (i, 1) *a*) nothing is gained by retention of A (or A') in (1), (6) or (7). So, *with non-dimensional significance for r, z and* ϕ, we may say that our problem is to find a solution of (5) which on a specified boundary has values calculated from

$$\phi = -\int r^2 \,.\, \widehat{\nu\theta}\, ds = \int r^2 (\widehat{r\theta}\, dz - \widehat{\theta z}\, dr), \qquad (6)\ bis$$

when (in any units) $\widehat{\nu\theta}$, $\widehat{r\theta}$, $\widehat{\theta z}$ denote *actual* intensities of stress. (The lower limit of integration is immaterial: it will be convenient to take it on the axis Oz.) When ϕ has been determined, the stress at every point can be calculated from

$$\widehat{r\theta} = \frac{1}{r^2}\frac{\partial \phi}{\partial z}, \quad \widehat{\theta z} = -\frac{1}{r^2}\frac{\partial \phi}{\partial r}, \qquad (1)\ \mathrm{A}$$

r and z again having non-dimensional significance.

Evaluation of ϕ on the basis of (6), and the subsequent computation of stresses on the basis of (1) A, present no difficulty when differentials are replaced by their finite-difference approximations. The governing equation (5) is replaced by

$$\mathbf{F}_0 = \Sigma_{a,4}(\phi) - 4\phi_0 - \frac{3}{2}\frac{a}{r}(\phi_2 - \phi_4) = 0, \qquad (25)$$

in which 0, 1, 2, 3, 4 have the same significance as in (23). (The term a^2 in that equation, which corresponds with the right-hand side of (2), is now suppressed.)

Example 2. Torsion of a shaft with solid collar

17. In illustration we have treated the problem discussed by Thom & Orr (§ 15), namely, a long shaft of radius 8 units carrying a collar, of length 6 and radius 17 units, which is joined by a fillet of radius 1 unit (see small sketch in figure 5). The computations (by D. N. de G. A.) were carried to a rather unusually fine net (having 4 mesh sides to 1 unit in figure 5). This would have had illegible numerals if reduced sufficiently for reproduction here: accordingly figure 5 summarizes results on a net having 1 mesh side to 1 unit, and figure 6 reproduces a portion of the finest net to show the variation of ϕ in the neighbourhood of the fillet. In both diagrams contours of ϕ are drawn to show the direction of the resultant shear stress.

Only one point in the computations calls for notice here. On cross-sections remote from the collar, the principle of Saint Venant shows that the stress will have its well-known distribution for a cylindrical shaft, i.e. that $\widehat{\theta z}$ will be proportional to r and consequently ϕ to r^4. It is therefore desirable to start by assuming that $\phi \propto r^4$ everywhere, then use the relaxation method to correct this assumption in way of the collar; for thereby the labour of computation is restricted to a relatively small region. *But in adopting this procedure we must take account of intrinsic errors in our finite-difference approximation* (25). There the assumption $\partial\phi/\partial z = 0$ means that $\phi_3 = \phi_1 = \phi_0$, so the typical condition for zero residual force reduces to

$$\mathbf{F}_0 = \phi_2 + \phi_4 - 2\phi_0 - \frac{3}{2}\frac{a}{r}(\phi_2 - \phi_4) = 0, \qquad (26)$$

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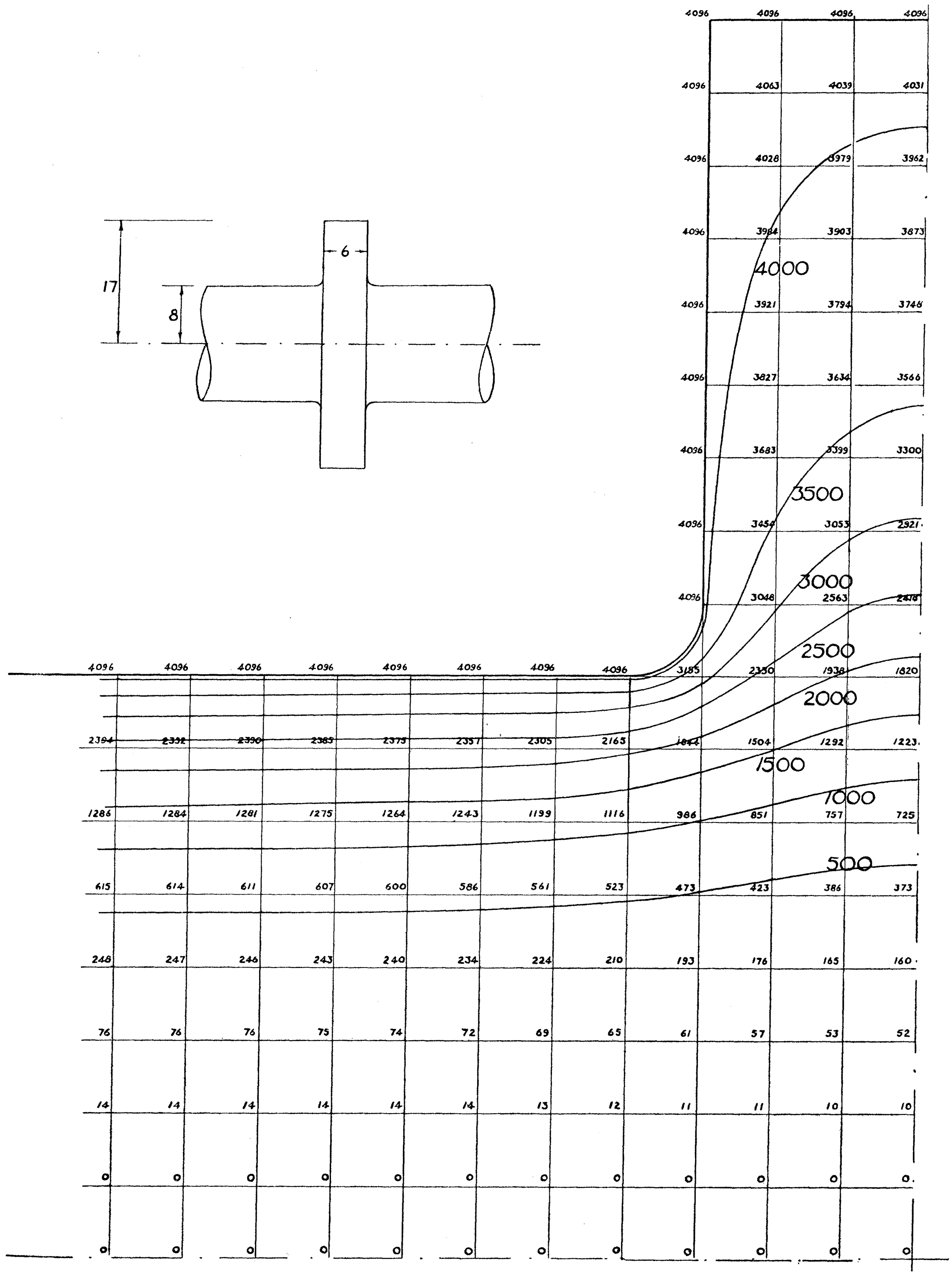

FIGURE 5

and this relation must be satisfied at sections far from the fillet. From it, assuming ϕ to be zero when $r = 0$ and to have the value $4096\,(=8^4)$ when $r = 8$ units, we deduce (for $a = \frac{1}{2}$ unit)† that when

$r =$	1,	2,	3,	4,	5,	6,	7,	8 units
$\phi =$	0,	14,	76,	248,	615,	1286,	2394,	4096

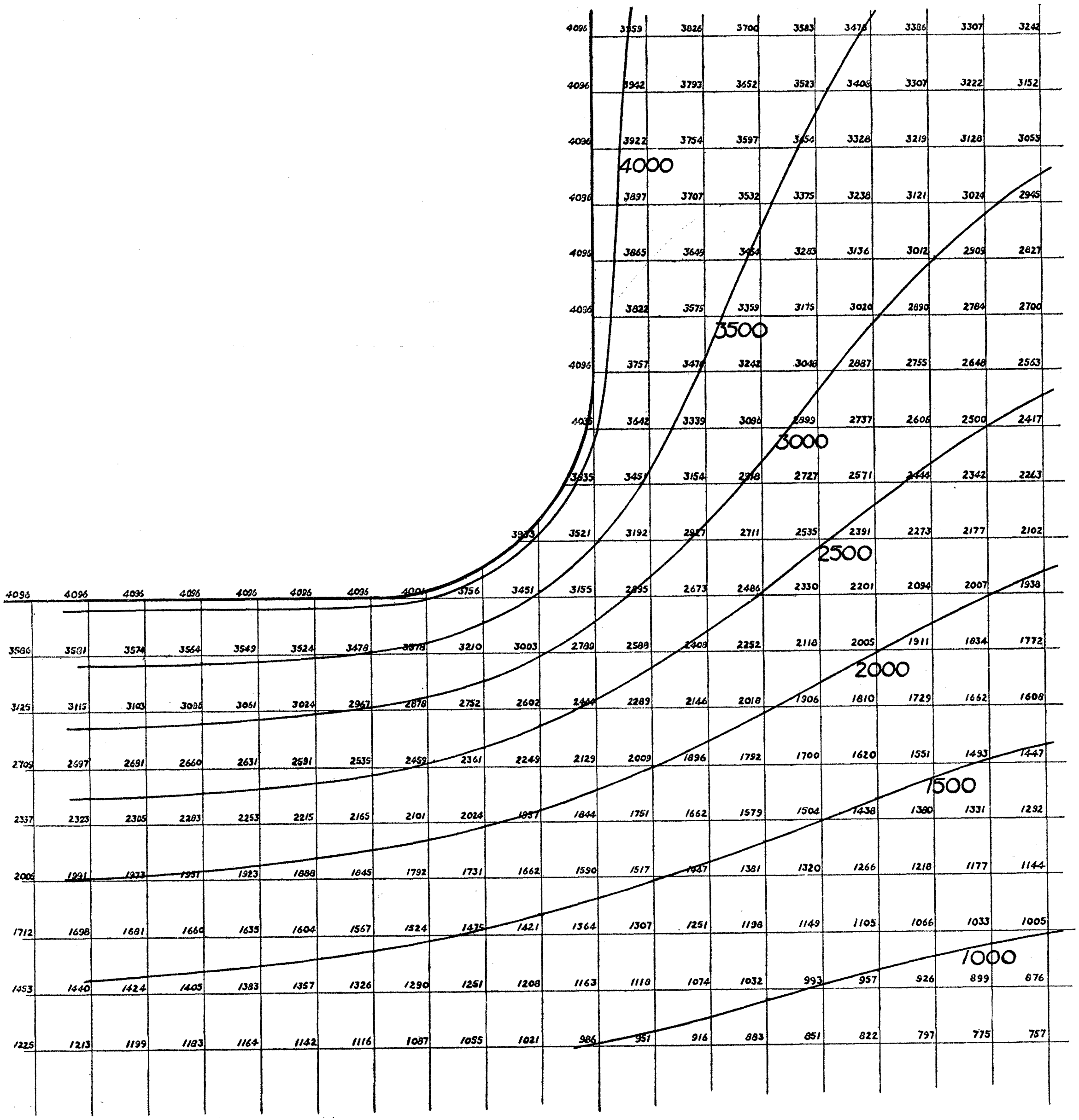

Figure 6

† A different series of numbers will be obtained from (26) for every chosen value of a. In figure 5 the recorded solution has been transferred from a larger working diagram ($a = \frac{1}{2}$).

When ϕ has this distribution at great distances, it must also have the value 4096 all along the stress-free boundary, and it must vanish at every point on Oz. The value 4096 on the stress-free boundary was the value taken by Thom & Orr.

18. The fine-net calculations recorded in figure 6 were made in order to obtain a reliable estimate of the stress-intensification at the fillet. Figure 7 compares our results with those of Thom & Orr for the variation of the resultant shear stress along the boundary. Some differences are apparent—as was to be expected in view of the relatively coarse net employed by those authors (one mesh-side of their finest net corresponded with two of the net in figure 6).

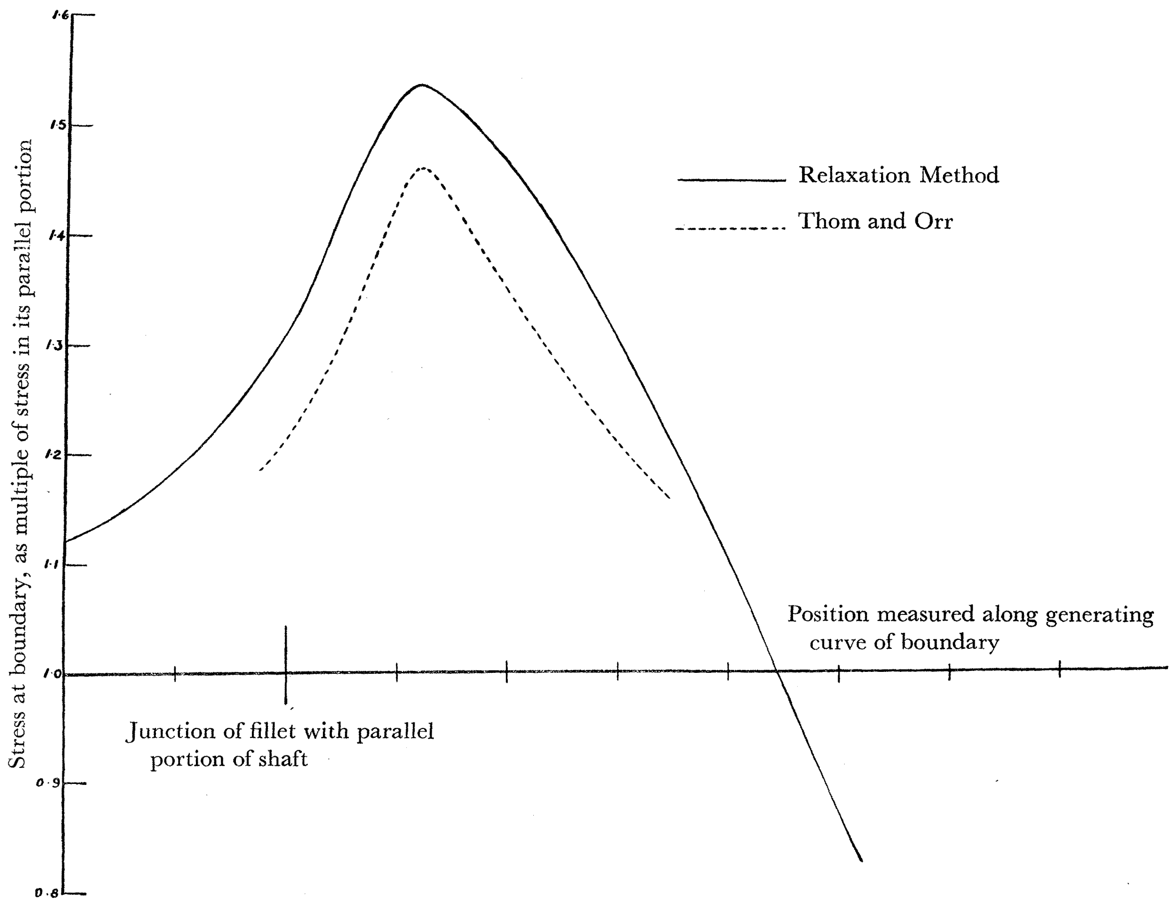

FIGURE 7

Problem (i, 2) *a: Axially symmetrical strain in a complete solid of revolution*

19. The wanted function λ is governed by equation (9) and is subject to the boundary conditions (11) of § 5, in which ψ and χ are related with λ by (10), therefore satisfy (12). All of these equations retain their forms when we substitute

$$r' \text{ for } r/L, \quad z' \text{ for } z/L, \quad \lambda' \text{ for } \lambda/L^4, \quad \Theta', \chi', \psi' \text{ for } (\Theta, \chi, \psi)/L^2, \tag{27}$$

L denoting some representative dimension of the cross-section and ν', s' being written for $(\nu, s)/L$. With dashed (i.e. 'non-dimensional') significance for r, ν, s, ψ, χ in (11), $\widehat{\nu r}$ and $\widehat{\nu z}$ still represent actual intensities of surface traction.

As in §12 we adopt this non-dimensional interpretation of the symbols *without actually inserting dashes in the equations.*

20. Since different 'patterns' must in any event be used at different radii, no advantage is gained by a use of nets of square as contrasted with rectangular mesh. With mesh-side a in the direction of z, b in the direction of r (figure 2), the finite-difference approximations to the first and second of (12) are respectively

$$\left.\begin{aligned}\mathbf{F}_\chi &= \chi_1+\chi_3-2\chi_0+\left(\frac{a}{b}\right)^2(\chi_2+\chi_4-2\chi_0)-\frac{a^2}{2rb}(\chi_2-\chi_4)=0\\ \text{and}\quad \mathbf{F}_\psi &= \psi_1+\psi_3-2\psi_0+\left(\frac{a}{b}\right)^2(\psi_2+\psi_4-2\psi_0)-\frac{a^2}{2rb}(\psi_2-\psi_4)+2\chi_0-\chi_1-\chi_3=0,\end{aligned}\right\} \tag{28}$$

according to (21), when the points 0, 1, 2, 3, 4 are arranged as in figure 2. (The mesh-sides a and b, like a in §13, are now regarded as chosen multiples of the representative dimension L, so that both of the factors (a/b) and $(a^2/2rb)$, in (28), are purely numerical.) From these expressions for $\mathbf{F}_\chi$ and $\mathbf{F}_\psi$ their initial values, and appropriate 'patterns' for the relaxation process, may be derived in the manner of §13.

Of the boundary conditions (11), the second may be integrated to give

$$\psi = -\int r \,.\, \widehat{\nu z}\, ds, \tag{29}$$

which fixes ψ on the boundary. (The lower limit of integration is immaterial, and may conveniently be taken on the axis Oz.) The first, which may be written in the equivalent form

$$\cos(r,\nu)\left[\frac{1}{r}\frac{\partial}{\partial r}-\frac{1-\sigma}{r^2}\right]\chi = \widehat{\nu r}-\cos(r,\nu)\left[\frac{1}{r}\frac{\partial}{\partial r}-\frac{1}{r^2}\right]\psi-\frac{\sin(r,\nu)}{r}\frac{\partial\psi}{\partial z}, \tag{30}$$

must be replaced by its finite-difference approximation.

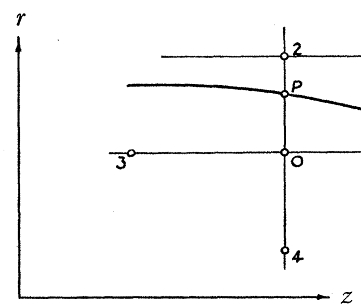

FIGURE 8

21. Owing to the possibility of 'irregular stars' of which the interrupted arms may have either the z- or r-direction or both, general treatment of (30) is not worth while

and every case calls for special consideration. No great difficulty is confronted when the only interrupted arm has the direction Or (figure 8): ψ_P has been determined from (29), and χ_P can be expressed (approximately) in terms of χ_0 and χ_2 (2 being a 'fictitious point') by the formula

$$\chi_P = \eta \cdot \chi_2 + (1-\eta)\,\chi_0, \tag{i}$$

in which ηb denotes the length of the interrupted arm OP; to the same approximation we may write

$$b\left(\frac{\partial \chi}{\partial r}\right)_P = \chi_2 - \chi_0, \quad \eta b\left(\frac{\partial \psi}{\partial r}\right)_P = \psi_P - \psi_O, \tag{ii}$$

and $(\partial\psi/\partial z)_P$ can be related with $(\partial\psi/\partial r)_P$ by the second of (11), written in the equivalent form

$$\cos(r,\nu)\frac{\partial\psi}{\partial z} - r\,.\,\widehat{\nu z} = \sin(r,\nu)\frac{\partial\psi}{\partial r}; \tag{11) B}$$

so, for the point P in figure 8, the required approximation to (30) is

$$\cos(r,\nu)\left[\frac{r_P}{b}(\chi_2-\chi_0) - (1-\sigma)\{\eta\chi_2 + (1-\eta)\,\chi_0\}\right]$$
$$= r_P^2(\widehat{\nu r})_P - \cos(r,\nu)\left[\frac{r_P}{\eta b}(\psi_P - \psi_O) - \psi_P\right] - \tan(r,\nu)\left[r_P^2(\widehat{\nu z})_P + \sin(r,\nu)\frac{r_P}{\eta b}(\psi_P-\psi_O)\right], \tag{31}$$

in which r_P, η, ψ_P, $(\widehat{\nu r})_P$, $(\widehat{\nu z})_P$ are known. Eliminating χ_2 between (28) and (31) we have relations from which special relaxation patterns can be deduced for a point 0 close to the boundary.

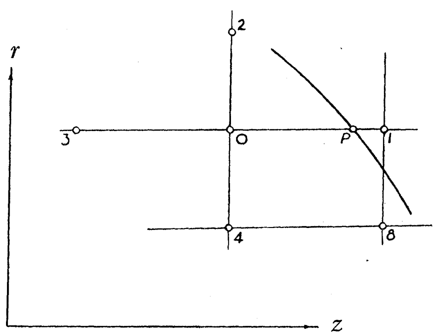

FIGURE 9a

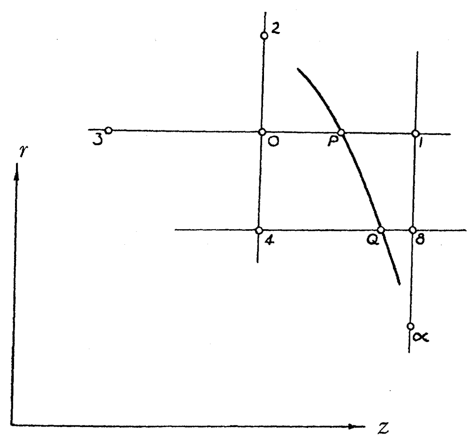

FIGURE 9b

22. The case is harder when the star has an interrupted arm parallel to Oz. There are two possibilities, according as 8 lies within the boundary (figure 9a) or outside it (figure 9b).

In relation to figure 9a we have, corresponding with (i),

$$\chi_P = \xi\chi_1 + (1-\xi)\,\chi_0, \tag{iii}$$

ξa denoting the length of the interrupted arm OP and 1 being a 'fictitious point'; and corresponding with the second of (ii)

$$\xi a\left(\frac{\partial\psi}{\partial z}\right)_P = \psi_P - \psi_O; \tag{iv}$$

also $(\partial\psi/\partial r)_P$ is related with $(\partial\psi/\partial z)_P$ by (11) B as before: but now in place of the first of (ii) we must employ

$$b\left(\frac{\partial\chi}{\partial r}\right)_P = \xi(\chi_1 - \chi_8) + (1-\xi)(\chi_0 - \chi_4), \tag{v}$$

as the closest approximation which is practicable. Substituting in (30) we have an equation in which r_0, ξ, ψ_P, $(\widehat{vr})_P$, $(\widehat{vz})_P$ are known, and between this and (28) χ_1 may be eliminated in the manner of § 21.

When the point 8 is also 'fictitious' (figure 9 *b*), χ_8 has also to be eliminated, and it is difficult to obtain an independent expression for this quantity. The best procedure in this case would seem to be substitution of the equivalent relation

$$\vartheta^2\psi + \left[\frac{\partial^2}{\partial r^2} - \frac{1}{r}\frac{\partial}{\partial r}\right]\chi = 0 \tag{12) A}$$

for the second of (12), leading to a corresponding substitution of

$$\begin{aligned}\mathbf{F}_\psi = \psi_1+\psi_3-2\psi_0+\left(\frac{a}{b}\right)^2(\psi_2+\psi_4-2\psi_0)-\frac{a^2}{2rb}(\psi_2-\psi_4)\\ +\left(\frac{a}{b}\right)^2(\chi_2+\chi_4-2\chi_0)-\frac{a^2}{2rb}(\chi_2-\chi_4)=0\end{aligned} \tag{28) A}$$

for the second of (28). Thereby we avoid the difficulty associated with that governing equation; but χ_8 still appears in the first of (28), and it can be eliminated only by a use of the boundary condition at Q and a consequent introduction of χ_α. If α is an internal point, all fictitious values will have been eliminated: if α is an external point, it can in turn (theoretically) be eliminated, and the process can be continued down the line 1–8–α until eventually the boundary is crossed; but in practice little benefit would be gained by advancing beyond α. Every case calls for special devices *ad hoc*, and no problem, in our experience, has presented great difficulty. Sometimes a sufficiently good approximation is to replace $(\partial\chi/\partial r)_P$ by $(\partial\chi/\partial r)_O$: in other cases ($\psi_P$ being always known) ψ_O can be obtained by interpolation and (12) A can then be interpreted as the equation governing χ.

Example 3. *Straight tube loaded with a belt of uniform pressure*

23. Binnie (1941) has treated, by extension of a method due to Filon (1902), the case of a short thick tube which is stressed by a belt of uniform pressure applied to its external surface (figure 10). The radial and longitudinal displacements are expressed as two series of product functions typified by $I_1(kr)\cos kz$, and the coefficients in the series are adjusted so as to make the corresponding expression for $\widehat{rr}$ reduce at the cylindrical boundaries to the Fourier series for the specified pressures. The labour entailed is

considerable, and prohibits an exact reproduction of this loading: in Binnie's paper it was reproduced to the accuracy shown by figure 11, in which the dotted curve gives the accepted approximation to the belt of uniform pressure, the full-line curve gives

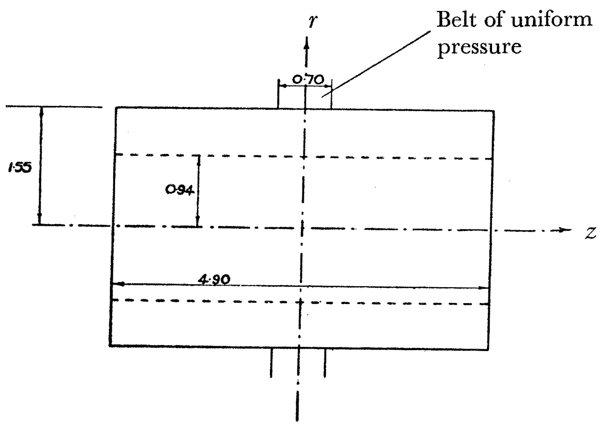

FIGURE 10

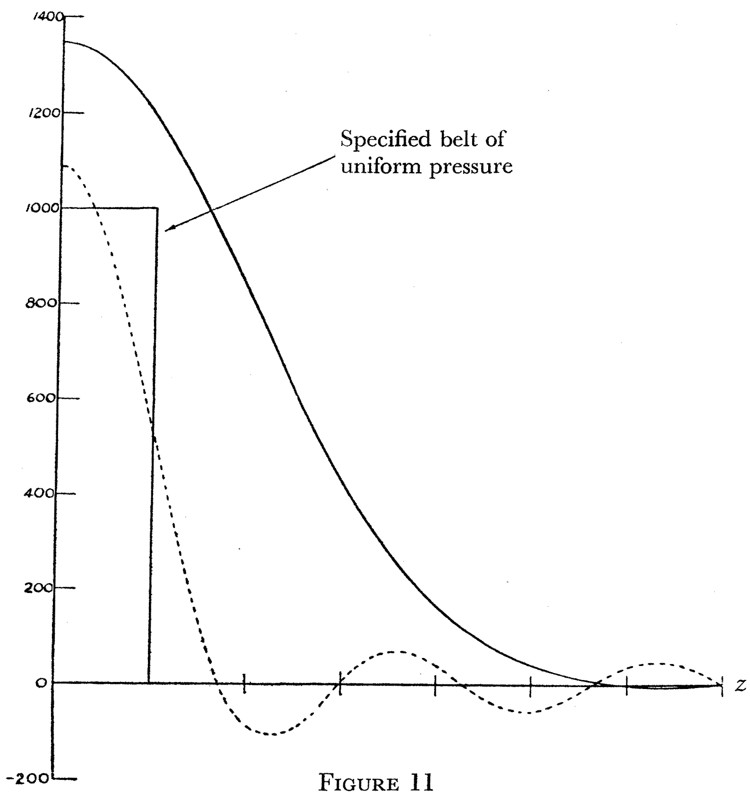

FIGURE 11

the longitudinal variation of $\widehat{\theta\theta}$ at the inner surface. Strictly, therefore, Binnie's full-line curve relates to a loading represented by his dotted curve: on that understanding, his work provides a test case whereby the accuracy of our methods may be assessed.

We now investigate this test case, and later (§ 25) we shall examine the order of the inaccuracy which is entailed by Binnie's incomplete reproduction of the specified

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loading. To make our solution strictly comparable, *in this instance we assume (with Binnie) that* σ *has the value* 0·25.

24. Figure 10 is an exact reproduction of Binnie's figure 1, except that our half-length (2·45 units) is a 'rounding off' of his half-length ($\frac{1}{2}\pi \times$ external radius $= 0{\cdot}775\pi = 2{\cdot}435$ units). A rectangular net was employed, having 14 mesh-sides in the half-length and 4 mesh-sides in the tube thickness: i.e. the assumed value of a was 0·175 unit, that of b was 0·1525 unit. Since the boundary was rectangular, this example did not entail the complexity of 'irregular stars' (§§ 21–2).

a/b having the value $70/61 = 1{\cdot}147_5$, equations (28) of § 20 become in this instance

$$\left.\begin{aligned} \mathbf{F}_\chi &= \chi_1+\chi_3-2\chi_0+1{\cdot}3168(\chi_2+\chi_4-2\chi_0)-\frac{0{\cdot}10041}{r}(\chi_2-\chi_4)=0,\\ \mathbf{F}_\psi &= \psi_1+\psi_3-2\psi_0+1{\cdot}3168(\psi_2+\psi_4-2\psi_0)-\frac{0{\cdot}10041}{r}(\psi_2-\psi_4)+2\chi_0-\chi_1-\chi_3=0,\end{aligned}\right\} \quad (32)$$

and the boundary conditions (after elimination of 'fictitious points' in the manner of §§ 21–2) give relations as under:

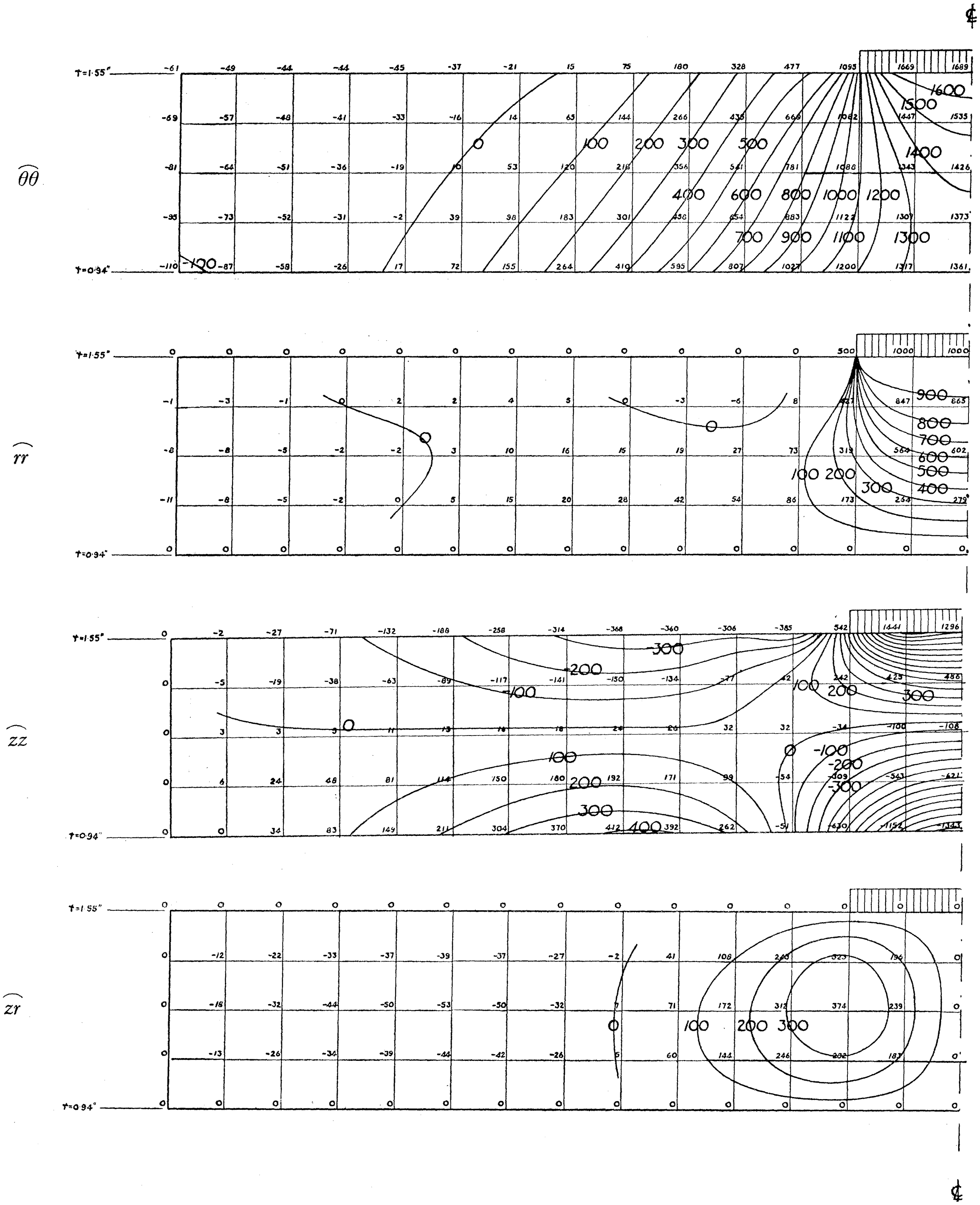
$$\left.\begin{aligned} &\textit{at the internal surface } (r = 0{\cdot}94 \text{ unit})\\ &\qquad \mathbf{F}_\chi = 2(\chi_2+\psi_2)-2{\cdot}2631\chi_0 = 0,\\ &\textit{at the external surface } (r = 1{\cdot}55 \text{ units})\\ &\qquad \mathbf{F}_\chi = 2(\chi_4+\psi_4)-1{\cdot}8597\chi_0+0{\cdot}4495\widehat{rr}_0 = 0,\\ &\textit{at the plane end } (z = -2{\cdot}45 \text{ units})\\ &\qquad \mathbf{F}_\chi = 1{\cdot}3168(\chi_2+\chi_4-2\chi_0)-\frac{0{\cdot}10041}{r}(\chi_2-\chi_4)+2\psi_1 = 0.\end{aligned}\right\} \quad (33)$$

In all of (32) and (33) the points 0, 1, 2, 3, 4 are assumed to have the arrangement shown in figure 2. Their representation in 'patterns', and the use of these to liquidate computed values of residual 'forces', follow on the lines of problems treated earlier.

$\widehat{vz}$ being zero everywhere, according to (29) ψ has a constant value on the boundary, and without loss of generality the constant can be taken as zero (cf. § 5).

25. Figure 12 presents the results of computation (by D. N. de G. A.), in diagrams giving nodal values and contours of the four non-zero stress components $\widehat{rr}$, $\widehat{\theta\theta}$, $\widehat{zz}$, $\widehat{zr}$. Figure 13 gives, for comparison with Binnie's results, the longitudinal variation of $\widehat{\theta\theta}$ at the inner surface. Both diagrams relate to the specified belt of uniform pressure (its intensity was taken as 1000), but a third curve (dotted) in figure 13 indicates by comparison the order—according to our computation—of the errors entailed by Binnie's incomplete representation of the specified loading (cf. § 23).

The latter are seen to be small, and over most of the length our values of $\widehat{\theta\theta}$ are in close agreement with Binnie's: there are, however, significant discrepancies close to



$\widehat{\theta\theta}$
$\widehat{rr}$
$\widehat{zz}$
$\widehat{zr}$
r = 1·55"
r = 0·94"
1600 1500 1400 1300 1200 1100 1000 900 800 700 600 500 400 300 200 100 0 -100
900 800 700 600 500 400 300 200 100 0
-300 -200 -100 0 100 200 300 400 -100 -200 -300
100 200 300 0

FIGURE 12

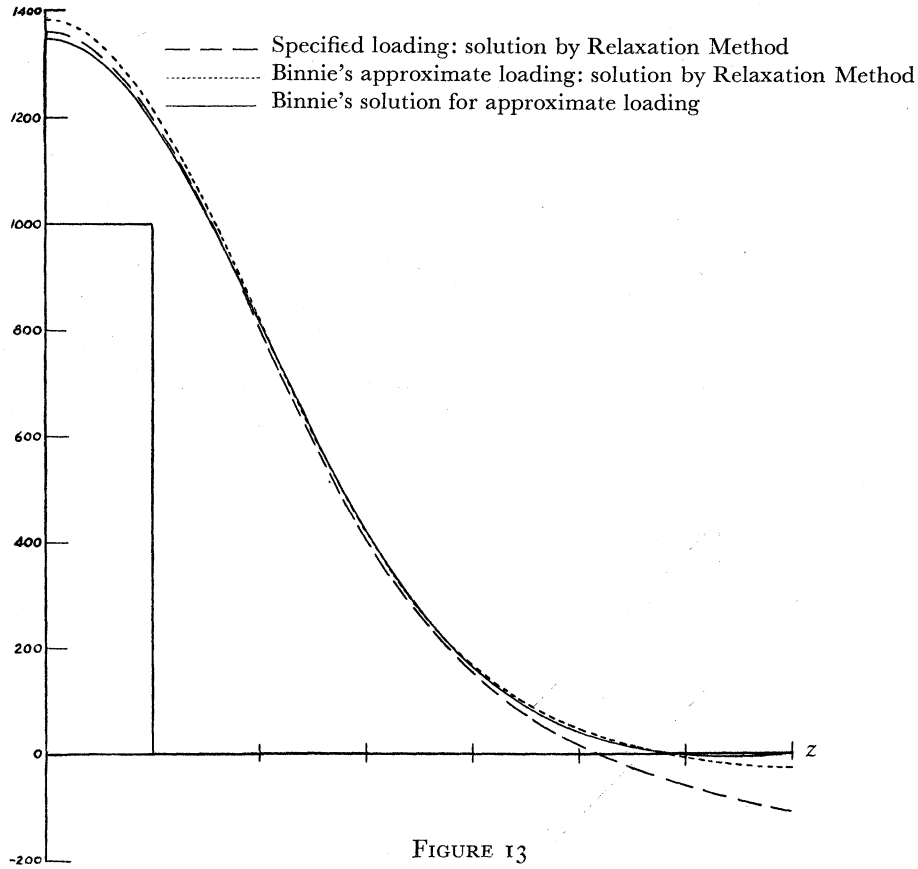


FIGURE 13

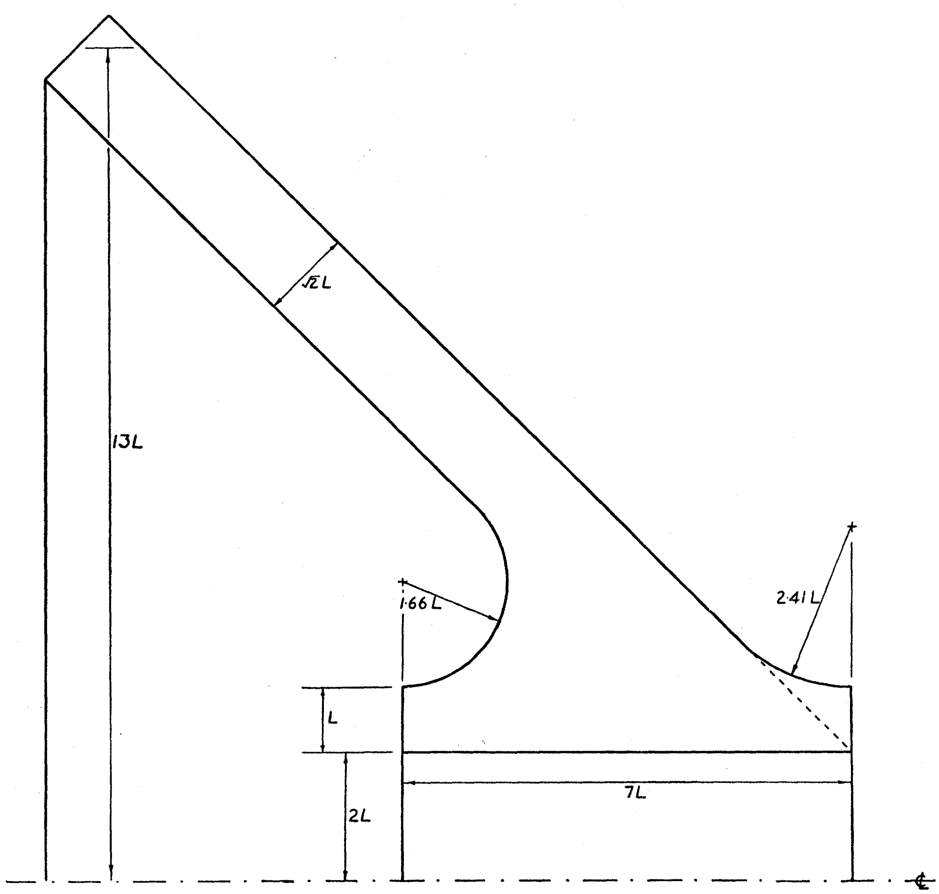


FIGURE 14

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the free ends. Binnie, by using a finite series of Fourier terms for $\widehat{\theta\theta}$, in effect compels that stress component to vanish at either end: our treatment indicates that in fact it does not vanish.

Example 4. *Centrifugal stresses in a conical rotor*

26. To illustrate the extended theory of §§ 6–8 (Section I) rotational stresses were computed (by D. N. de G. A.) for the conical rotor shown in figure 14. Very little further discussion of method is needed, since (§ 8) the stress function λ satisfies (9) as before and the boundary conditions (11) are altered only by an addition of specified terms to their left-hand sides: this means that §§ 19–22 hold without substantial modification.

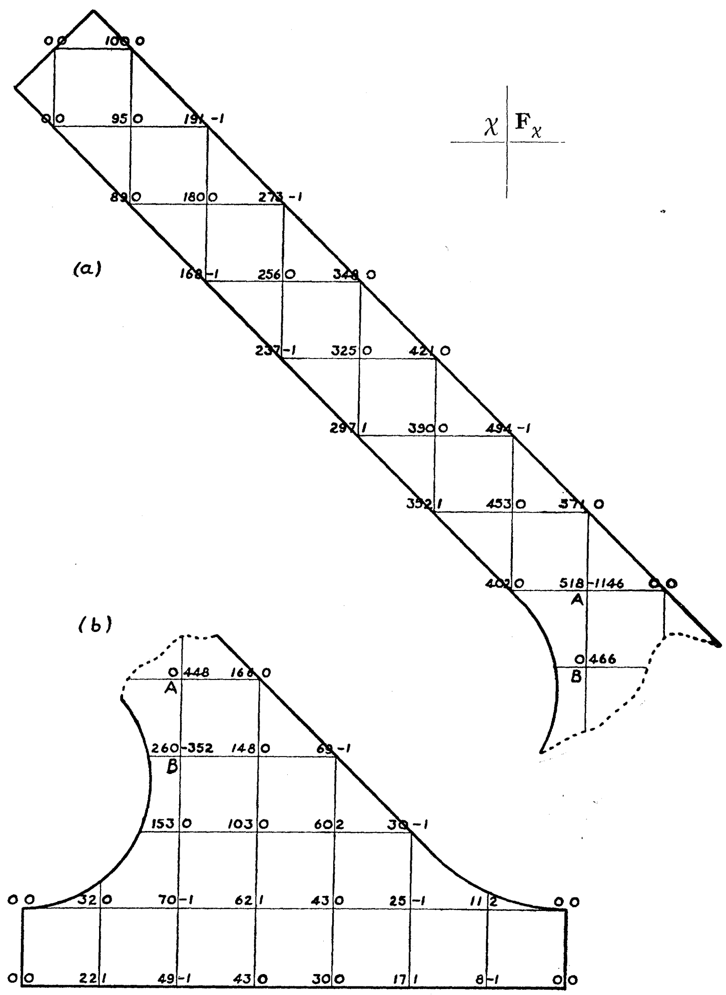


FIGURE 15

It was shown in § 19 that $\widehat{\nu r}$, $\widehat{\nu z}$ still represent *actual* intensities of surface traction when r, ν, s, ψ, χ, in (11), have 'non-dimensional' significance. The same will be true of (11) A, except that with this significance for r and z the acceleration terms $\rho\omega^2r^2$, $\rho f_z z$

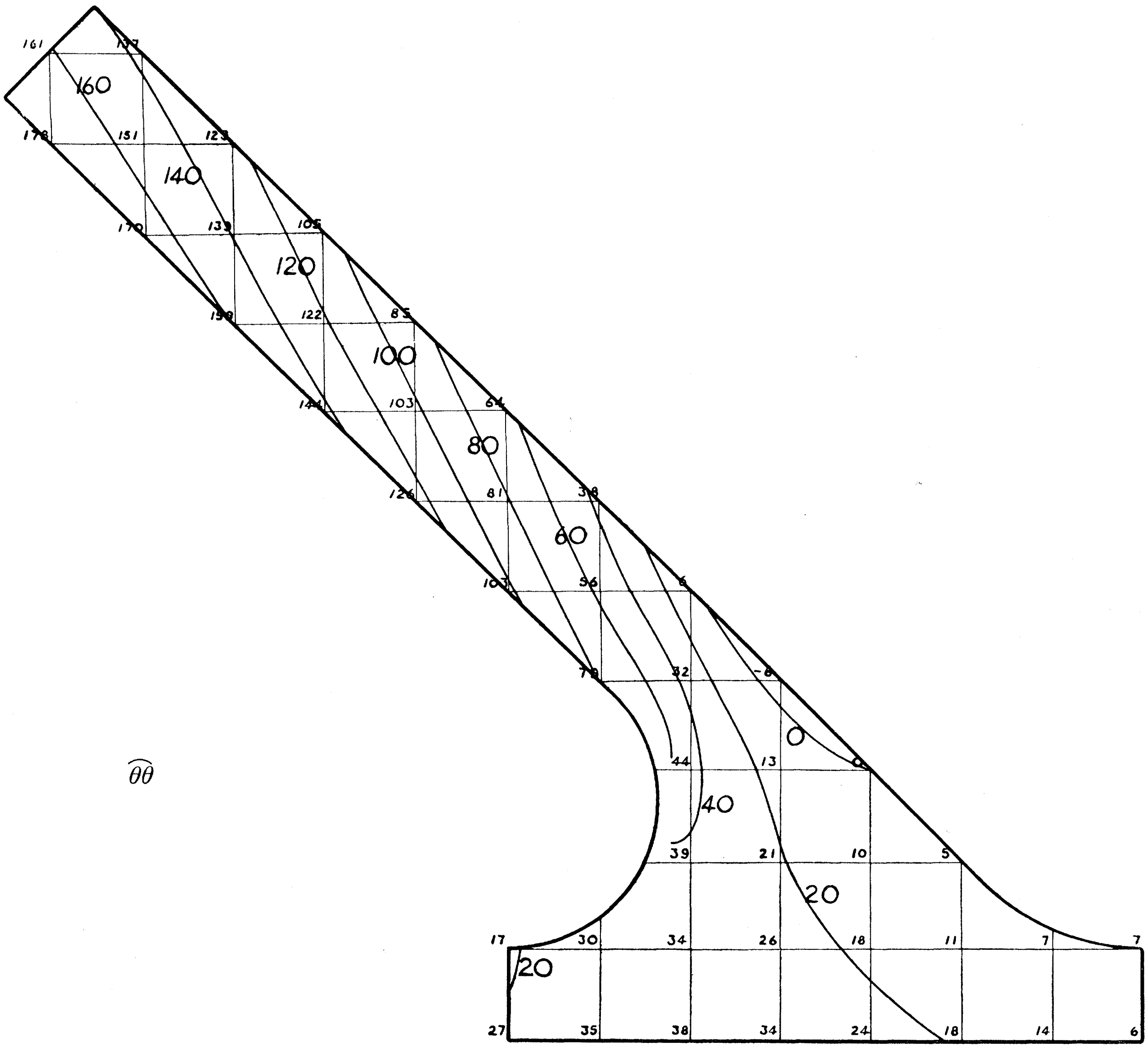


FIGURE 16*a*

must be replaced by $\rho\omega^2L^2r^2$, $\rho f_z Lz$ respectively. In this example no boundary tractions are operative and there is no longitudinal acceleration,—i.e.

$$\widehat{\nu r} = \widehat{\nu z} = f_z = 0;$$

consequently ψ and χ are directly proportional to $\rho\omega^2L^2$, and by taking that quantity as unity we shall obtain the stress components *as multiples of* $\rho\omega^2L^2$. The second of (11) A can be integrated to give (for $\sigma = 0{\cdot}3$)

$$\psi = -\tfrac{3}{56}r^4, \quad \text{on the boundary.} \tag{34}$$

Figure 14 exhibits our choice of the representative dimension L.

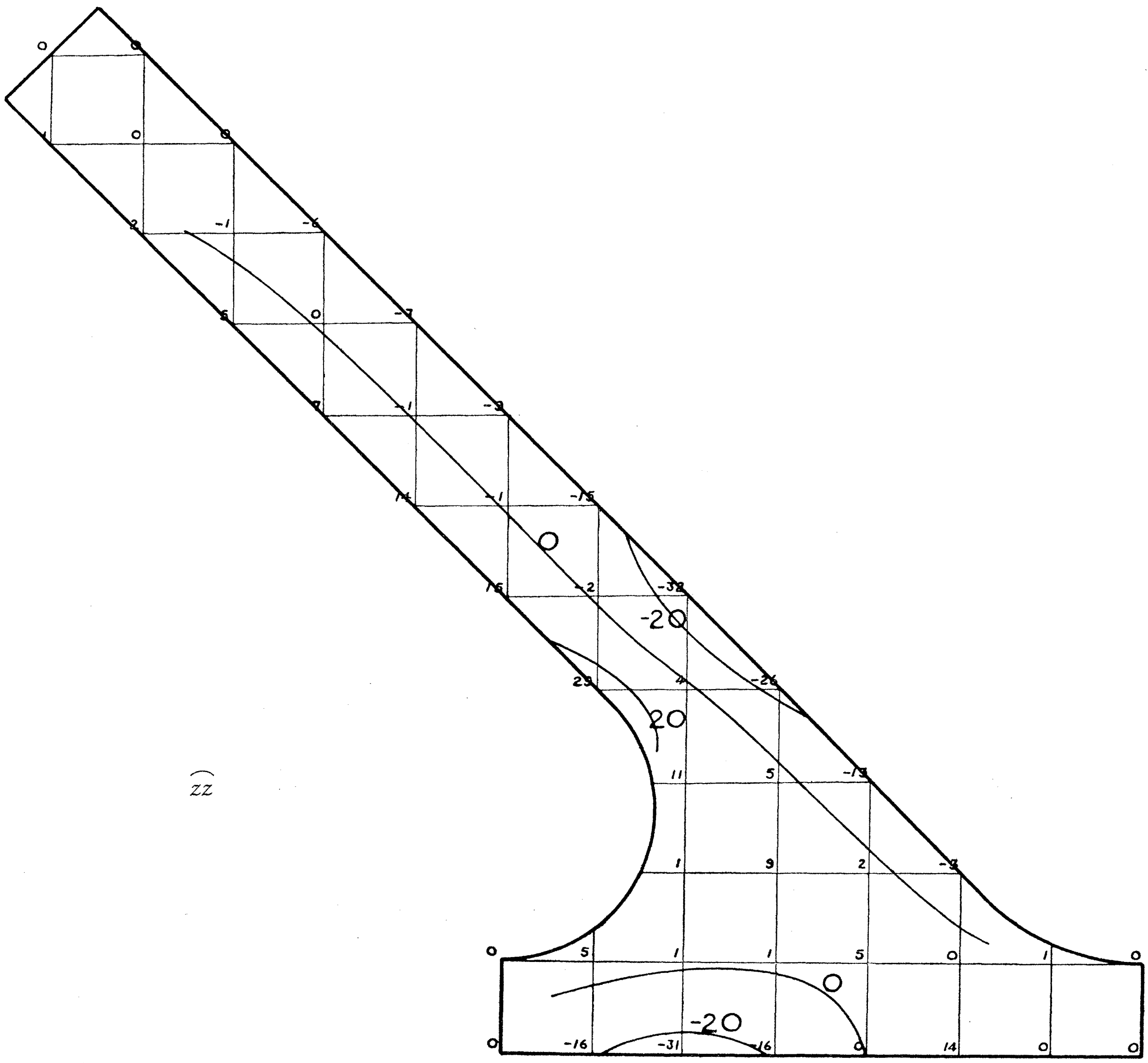
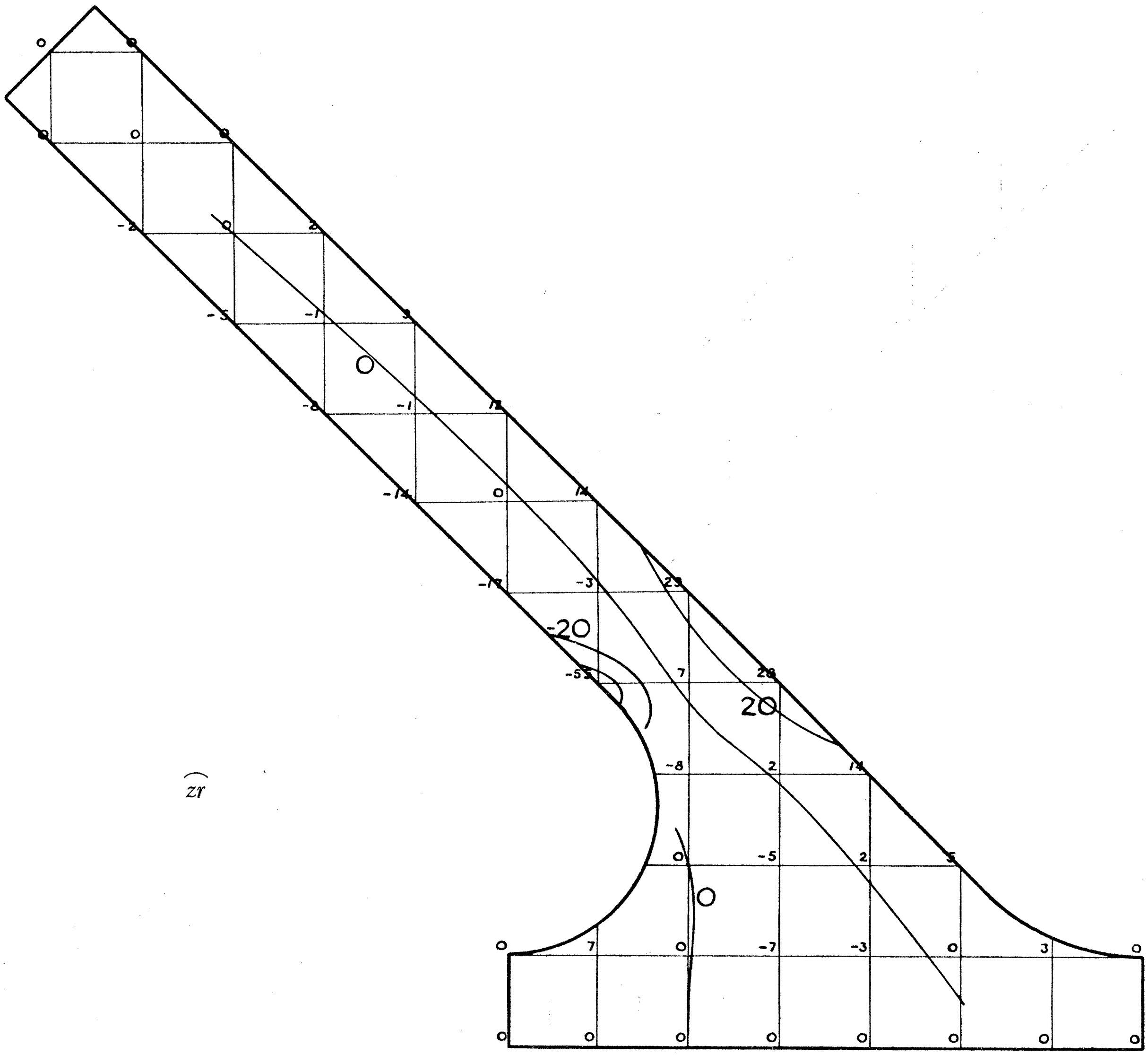


FIGURE 16 *b*

27. Two successive nets were employed, with square meshes of side L and $\frac{1}{2}L$. On the coarser net, relaxation was speeded by a use of the 'group displacements' shown in figure 15 *a*, *b*. These are 'partial solutions' entailing no boundary tractions and satisfying the governing equation (9)† everywhere except at A and B, where residual forces are an indication of 'singularities'. With their aid it is possible to eliminate the residuals which appear at A and B when a trial solution is 'built up' from the boundary to satisfy (9) exactly at all except those nodal points. This (for the boundary in question) is more accurate than the normal liquidation process.

Figures 16 exhibit our computed values of $\widehat{rr}$, $\widehat{\theta\theta}$, $\widehat{zz}$, $\widehat{zr}$—shown (for clarity) on nets of mesh side L, although derived on a net twice as fine. The diagrams are self-explanatory.

† More precisely, satisfying the finite-difference approximation to (9).



$\widehat{zr}$

FIGURE 16*c*

Problem (i, 2) *b: Flexural stresses in an incomplete tore*

28. The wanted function μ is governed by equation (14) and is subject to the boundary conditions (16) of § 9, in which χ and ψ are related with μ by (15). If now we write

$$\chi = (1+\sigma)\, Bz^2 + \phi, \tag{35}$$

equations (15) require that

$$\vartheta^2\phi = -2\sigma B, \quad \vartheta^2\psi = \frac{\partial^2\phi}{\partial z^2}, \tag{36}$$

and equivalent forms of (16) are

$$\psi = 0 \text{ (as before)}, \qquad \frac{\partial}{\partial \nu}\left(\frac{\psi}{r}\right) + \cos(r,\nu)\left\{\left[\frac{\partial}{\partial r}+\frac{\sigma}{r}\right]\left(\frac{\phi}{r}\right) + \sigma B\left(1+\sigma\frac{z^2}{r^2}\right)\right\} = 0. \quad \text{(16) A}$$

These are somewhat more convenient.

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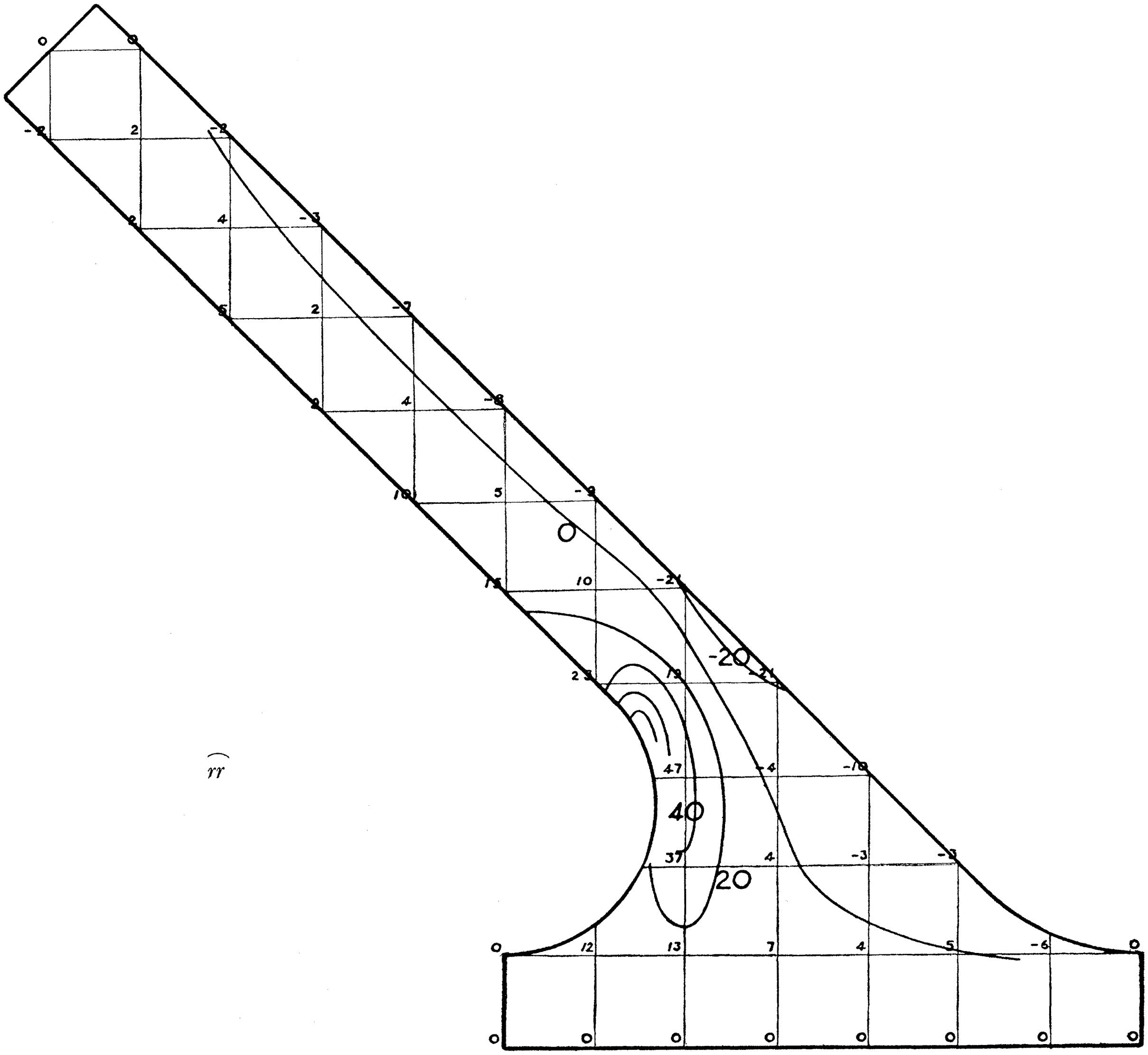

FIGURE 16*d*

All of (14), (15), (35), (36) and (16) A retain their forms when we substitute

$$r', z', v', s' \text{ for } (r, z, v, s)/L, \quad \mu' \text{ for } \mu/L^4, \quad \chi', \psi', \phi' \text{ for } (\chi, \psi, \phi)/L^2, \qquad (37)$$

L denoting some representative dimension of the cross-section. As before *we adopt this 'non-dimensional' interpretation of the symbols without actually inserting dashes in the equations.*

Example 5. *Curved bar of rectangular cross-section*

29. In illustration L. F. treated a curved bar (tore) of wide rectangular cross-section, having boundaries defined by

$$r = 4, \quad r = 6, \quad z = \pm 3 \text{ units.} \qquad (38)$$

(Since the computational aspects of this problem are closely identical with that of Problem (i, 2) *a*, and since little practical interest attaches to the bending of curved bars by pure couples, it did not seem worth while to introduce the complication of 'irregular' stars; and the wide rectangular section has theoretical interest as a test of the limitations of the solution of Golovin and others (Southwell 1942, § 33) for *narrow* rectangular sections.)

The symmetry of the problem allowed us to confine attention to one-half of the cross-section. A square net of side 0·5 unit was employed, and a value 100 was attached to σB in (16) A. Corresponding with (28) we have in this problem, as finite-difference approximations to the governing equations,

$$\left.\begin{aligned} \mathbf{F}_\phi &= \phi_1+\phi_2+\phi_3+\phi_4-4\phi_0-\frac{1}{4r}(\phi_2-\phi_4)+50=0,\\ \mathbf{F}_\psi &= \psi_1+\psi_2+\psi_3+\psi_4-4\psi_0-\frac{1}{4r}(\psi_2-\psi_4)-(\phi_1+\phi_3-2\phi_0)=0,\end{aligned}\right\} \quad (39)$$

and corresponding with (33), as boundary conditions obtained by elimination of 'fictitious points' in the manner of §§ 21–2,

$$\left.\begin{aligned} &\text{at the internal surface } (r=4)\\ &\qquad \mathbf{F}_\phi = 2(\phi_2+\psi_2)-2\cdot1859\phi_0+(475+7\cdot970z^2)=0,\\ &\text{at the external surface } (r=6),\\ &\qquad \mathbf{F}_\phi = 2(\phi_4+\psi_4)-1\cdot8930\phi_0-(525+4\cdot792z^2)=0,\\ &\text{and at the plane end } (z=-3),\\ &\qquad \mathbf{F}_\phi = \phi_2+\phi_4-2\phi_0-\frac{1}{4r}(\phi_2-\phi_4)+2\psi_0+50=0,\end{aligned}\right\} \quad (40)$$

together with $\psi = 0$ on all boundaries. (16) *bis*

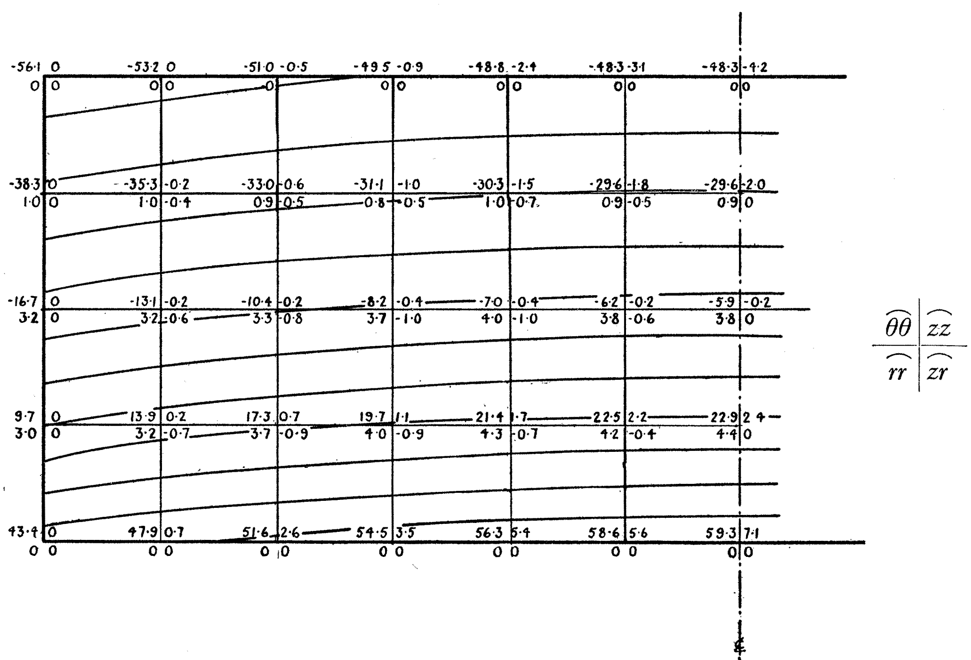

FIGURE 17. The curves are contours of $\widehat{\theta\theta}$.

30. The narrow-section solution (§ 29) would be exact if it were not for the term $\sigma z^2/r^2$ in (16) A, and this term is small; so it was not to be expected that the new solution would be widely different, and figure 17 shows that in fact all of $\widehat{rr}$, $\widehat{zz}$, $\widehat{zr}$ are small and $\widehat{\theta\theta}$ is very nearly independent of z.

Problem (ii). *Shearing and flexural stresses in a toroidal 'hook'*

31. The wanted functions are ϕ and ψ as defined by the governing equations (18) and by the boundary conditions (19) of § 10. All of these equations retain their forms when rendered 'non-dimensional' by the substitutions

$$r',\ z',\ \nu',\ s',\ Q' \ \text{for}\ (r,\ z,\ \nu,\ s,\ Q)/L, \quad \phi' \ \text{for}\ \phi/L^3, \quad \psi' \ \text{for}\ \psi/L^5, \tag{41}$$

L denoting some representative dimension of the cross-section; and in the resulting solution $\widehat{rr}$, $\widehat{\theta\theta}$, ..., etc., will still represent actual intensities of stress. As before, *we shall in computation adopt this non-dimensional interpretation of the symbols without actually inserting dashes in the equations.*

Mainly for the reason that three boundary conditions are imposed (instead of one or two), this problem is appreciably harder than any yet considered in this series, and even an approximate treatment must be expected to entail considerable labour. This is not a matter of much importance, because in practice hook sections are closely standardized, so that only one or two special cases will call for detailed treatment. Our aim has been to develop *iterative* methods of attack, one of which (Method 'A': largely due to D. N. de G. A. and L. F.) we now explain and illustrate.

Iterative method 'A'. Basic theory

32. If

$$\chi = \vartheta_3^2 \psi, \qquad r^3 \Omega = \frac{\partial^2 \psi}{\partial r\, \partial z}, \tag{42}$$

the governing equations (18) can be written as

$$\vartheta_3^2 \phi = \frac{\partial^2 \chi}{\partial z^2}, \qquad \vartheta_3^2 \chi = 2(Q + Az), \tag{18 A}$$

and the boundary conditions as

$$\left.\begin{aligned} \phi = \Omega = 0,& \\ \cos(r,\nu)\left\{Q(r^2+\sigma z^2) - \sigma A(zr^2 + \tfrac{1}{3}z^3) - 2\frac{\partial^2\psi}{\partial z^2} - \left[r\frac{\partial}{\partial r} - (3-\sigma)\right]\chi\right\} &= r\frac{\partial \phi}{\partial \nu}. \end{aligned}\right\} \tag{19 A bis}$$

From the definition of Ω we have

$$\left.\begin{aligned} \vartheta_{-3}^2 \Omega &= \frac{1}{r^3}\frac{\partial^2}{\partial r\,\partial z}\vartheta_3^2\psi = \frac{1}{r^3}\frac{\partial^2 \chi}{\partial r\,\partial z}, \\ \text{where} \qquad \vartheta_{-3}^2 &\equiv \frac{1}{r^3}\frac{\partial}{\partial r}\left(r^3\frac{\partial}{\partial r}\right) + \frac{\partial^2}{\partial z^2} \equiv \frac{\partial^2}{\partial r^2} + \frac{3}{r}\frac{\partial}{\partial r} + \frac{\partial^2}{\partial z^2}, \end{aligned}\right\} \tag{43}$$

also

$$\frac{\partial \Omega}{\partial z} = \frac{1}{r^3}\frac{\partial^3 \psi}{\partial r\,\partial z^2}, \quad \frac{\partial \Omega}{\partial r} = \frac{1}{r^3}\left[\vartheta_3^2 - \frac{\partial^2}{\partial z^2}\right]\frac{\partial \psi}{\partial z} = \frac{1}{r^3}\frac{\partial}{\partial z}\left(\chi - \frac{\partial^2 \psi}{\partial z^2}\right),$$

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and hence, at points on the boundary,

$$r^3 \frac{\partial \Omega}{\partial \nu} = \cos(r, \nu) \frac{\partial \chi}{\partial z} + \frac{\partial}{\partial s}\left(\frac{\partial^2 \psi}{\partial z^2}\right). \tag{44}$$

From this relation boundary values of $\partial^2\psi/\partial z^2$ could be determined if boundary values of $\partial\chi/\partial z$ and of $\partial\Omega/\partial\nu$ were known.

If, on the other hand, boundary values of $\partial^2\psi/\partial z^2$ were known, then equations (18) A, with the first and last of (19) A, would serve to determine ϕ and χ: they would present a problem similar to Problem (i, 2) *a* of § 5, where two functions χ and ψ, defined in (10), are determined by (12) combined with two boundary conditions (11). Having found that problem to be soluble by the methods outlined in §§ 20–25, we now make those methods the basis of an iterative attack, *starting from the assumption that* $\partial^2\psi/\partial z^2$ *is zero on the boundary.*

33. Simplifying the last of (19) A by this assumption, we can solve (18) A to obtain a first approximation (ϕ_1, χ_1, say) to ϕ and χ, and we can deduce a corresponding first approximation to Ω from (43) combined with the second of (19) A,—i.e. from

$$\left.\begin{aligned} &\vartheta^2_{-3}\Omega_1 = \frac{1}{r^3}\frac{\partial^2\chi_1}{\partial r\,\partial z} \\ &\text{combined with the boundary condition} \\ &\Omega_1 = 0. \end{aligned}\right\} \tag{45}$$

Then, having substituted Ω_1, χ_1 for Ω, χ in (44), we can from that equation deduce new values of $\partial^2\psi/\partial z^2$ on the boundary.

These values, denoted by $\partial^2\psi_2/\partial z^2$, can now be made the starting point of a second stage of computation. In this we seek corrections (ϕ_2, χ_2, say) to our first approximations ϕ_1, χ_1, using for that purpose equations derived from (18) A, viz.

$$\vartheta^2_3\phi_2 = \frac{\partial^2\chi_2}{\partial z^2}, \qquad \vartheta^2_3\chi_2 = 0, \tag{46}$$

combined with boundary conditions derived from the first and last of (19) A, viz.

$$\phi_2 = 0, \qquad \cos(r,\nu)\left\{2\frac{\partial^2\psi_2}{\partial z^2} + \left[r\frac{\partial}{\partial r} - (3-\sigma)\right]\chi_2\right\} + r\frac{\partial\phi_2}{\partial\nu} = 0; \tag{47}$$

then we deduce the corresponding correction to Ω (Ω_2, say) from equations similar in form to (45); and finally, substituting Ω_2, χ_2, ψ_2 for Ω, χ, ψ in (44), we integrate that equation to obtain new boundary values $\partial^2\psi_3/\partial z^2$ which become, similarly, the starting point of a third stage of computation. The whole process can be repeated indefinitely, and it may be terminated if and when the corrections prove to be negligible.

34. We then have distributions of ϕ, χ ($= \vartheta^2_3\psi$) and $r^3\Omega$ ($= \partial^2\psi/\partial r\,\partial z$), together with boundary values of $\partial^2\psi/\partial z^2$ from which, with the equation

$$\vartheta^2_3\frac{\partial^2\psi}{\partial z^2} = \frac{\partial^2\chi}{\partial z^2} \tag{48}$$

the distribution of $\partial^2\psi/\partial z^2$ can be computed. Using these results, we can compute all six of the stress components from their expressions (17), § 10.

Those expressions give (for a toroidal hook appropriately loaded) $\widehat{r\theta}$, $\widehat{\theta z}$ with omission of the factor $\cos\theta$, $\widehat{rr}$, $\widehat{\theta\theta}$, $\widehat{zz}$, $\widehat{zr}$ with omission of the factor $\sin\theta$, θ having the significance which is indicated by figure 18. That is to say, they give actual values of $\widehat{r\theta}$, $\widehat{\theta z}$ at the loaded section $\theta = 0$, actual values of $\widehat{rr}$, $\widehat{\theta\theta}$, $\widehat{zz}$, $\widehat{zr}$ at the section $\theta = \frac{1}{2}\pi$ in figure 18. (Cf. Southwell 1942, § 24 and figure 2, of which figure 18 is a reproduction.)

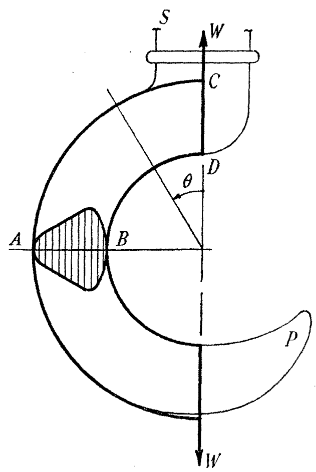

FIGURE 18

35. The argument of §§ 32–4 presumes that $\partial^2\psi/\partial z^2$ as derived from (44) will be single-valued. This follows from (43): for according to (44)

$$\oint \frac{\partial}{\partial s}\left(\frac{\partial^2\psi}{\partial z^2}\right) ds = \oint \left\{ r^3 \frac{\partial\Omega}{\partial\nu} - \cos(r,\nu)\frac{\partial\chi}{\partial z}\right\} ds$$

$$= \iint \left(r^3\,\vartheta^2_{-3}\,\Omega - \frac{\partial^2\chi}{\partial r\,\partial z}\right) dr\,dz \text{ for the whole cross-section (by Green's transformation),}$$

$$= 0 \text{ by (43).}$$

Moreover *the constant of integration is without importance*. For according to the last of (19) A a constant addition C to the boundary values of $\partial^2\psi/\partial z^2$ will entail a constant addition $\Delta\chi = 2C/(3-\sigma)$ to the boundary values of χ and therefore, by the second of (18) A, a constant addition $\Delta\chi$ *everywhere*: this will entail no change in ϕ as determined by the first of (18) A. Again, $\Delta\chi$ is compatible with a change $\Delta\psi$ in ψ which makes

$$\frac{\partial^2}{\partial z^2}\Delta\psi = C, \quad \textit{everywhere}, \tag{i}$$

provided that

$$r^3 \frac{\partial}{\partial r}\frac{1}{r^3}\frac{\partial}{\partial r}\Delta\psi = \Delta\chi - \frac{\partial^2}{\partial z^2}\Delta\psi$$

$$= C\left(\frac{2}{3-\sigma} - 1\right) = -\frac{1-\sigma}{3-\sigma}C, \tag{ii}$$

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and the relations (i) and (ii) are compatible with

$$\frac{\partial^2}{\partial r\,\partial z}\Delta\psi = 0, \tag{iii}$$

so $\Delta\psi$ entails no change in Ω as determined from (43). It follows that C will not affect $\widehat{zz}$, $\widehat{zr}$, $\widehat{\theta z}$ as given by (17); and since the consequent change in

$$2\frac{\partial^2\psi}{\partial z^2}-(3-\sigma)\,\vartheta_3^2\psi$$

is zero, neither will it affect $\widehat{rr}$, $\widehat{\theta\theta}$, $\widehat{r\theta}$ as given by (17).

Example 6. *Method 'A' applied to a hook of square cross-section*

36. To test the process L. F., with assistance from Miss Gillian Vaisey, has applied it to compute stresses in a hook of the square section bounded by

$$r = L, \quad r = 3L, \quad z = \pm L. \tag{49}$$

($2L$ is the side of the square.) A square net of mesh-side $a = \frac{1}{4}L$ was employed, and for this finite-difference approximations to the governing equations and to the boundary conditions were formulated in the manner of preceding sections.

The 'Q-solution' (cf. § 10) was studied separately from the 'A-solution', which has little practical importance but would be required in order to take account of loading applied otherwise than through the centre of the cross-section.

(1) '*Q-solution*' (*for centrally applied loading*)

37. First, with A made zero to give the case of centrally applied loading, Q (to which all stresses are proportional) was given an arbitrary value 800. No serious difficulty was confronted, and five stages of the process outlined in § 33 led to a solution which (since the last stage had yielded negligible corrections) could be deemed sufficiently exact. This is presented in figure 19, which records the accepted values of $\vartheta_3^2\psi$, $\partial^2\psi/\partial z^2$, Ω and ϕ.

From figure 19 the stress components $\widehat{r\theta}$, $\widehat{\theta\theta}$, $\widehat{\theta z}$ (which alone have practical importance) were computed (by G. V.) in the manner of § 34. Figure 20 records their values, and contours show the distribution of $\widehat{\theta\theta}$ (for comparison with the conclusions of an approximate treatment).

(2) '*A-solution*' (*for torsional loading*: cf. § 10)

38. This second problem (for its theoretical interest) was attacked by similar methods, Q being made zero and A (to which all stresses are proportional) being given an arbitrary value 2000. As was to be expected, more labour was entailed in this instance, six stages of the iterative process, with two (highly successful) syntheses, being needed to give the solution which is recorded in figure 21.

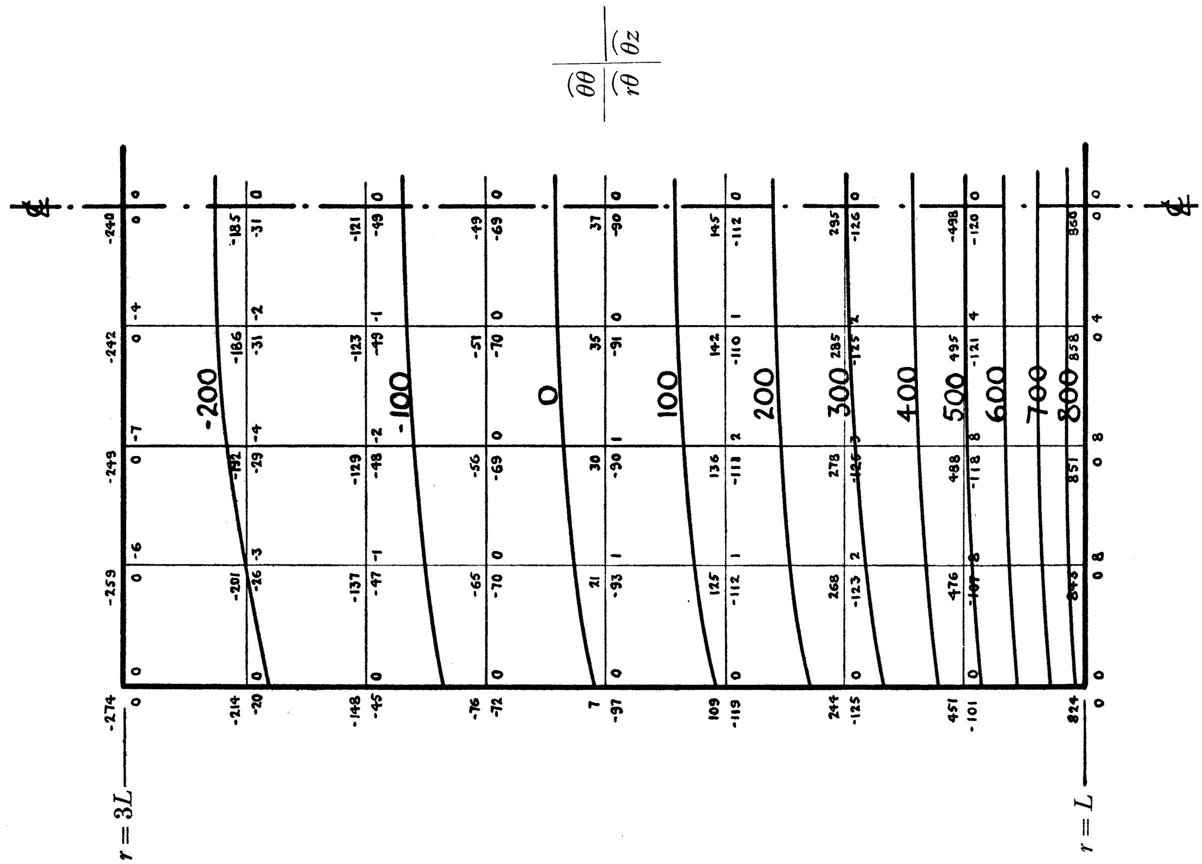

FIGURE 20. The curves are contours of $\widehat{\theta\theta}$.

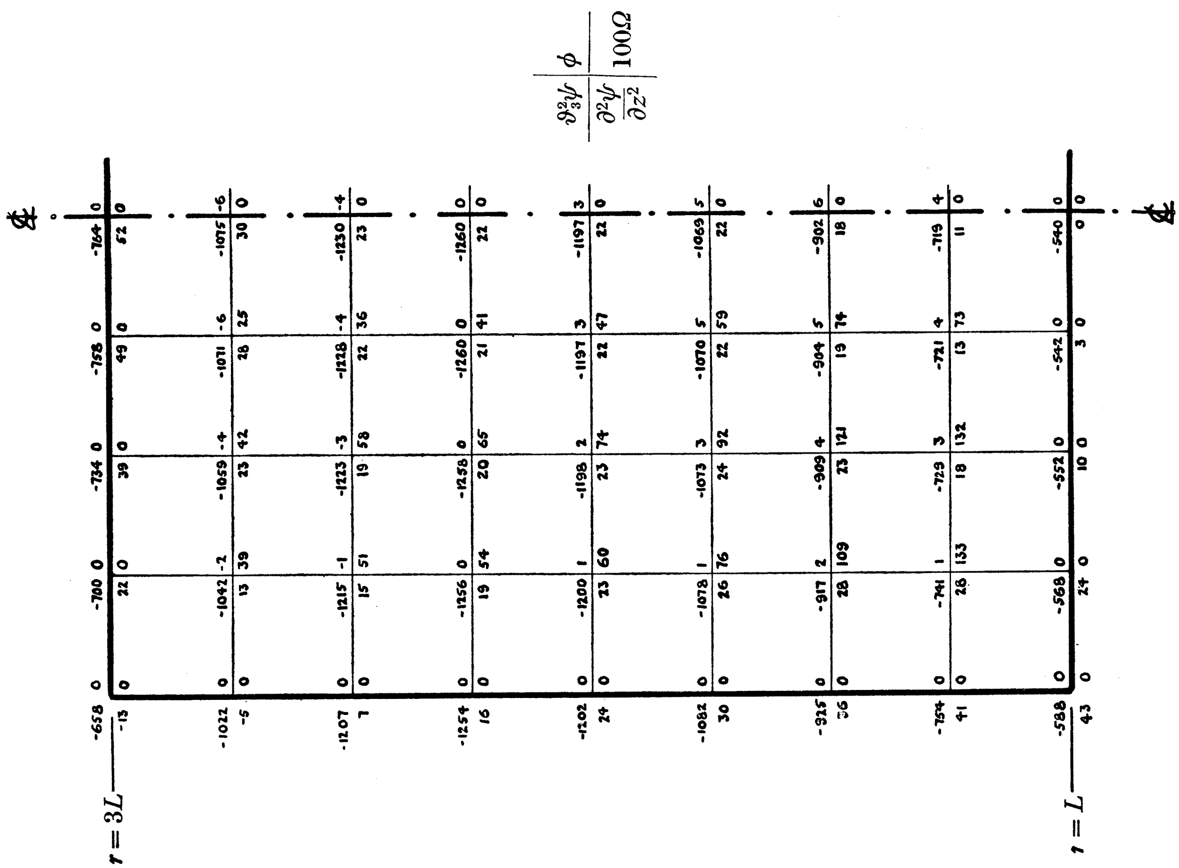

FIGURE 19

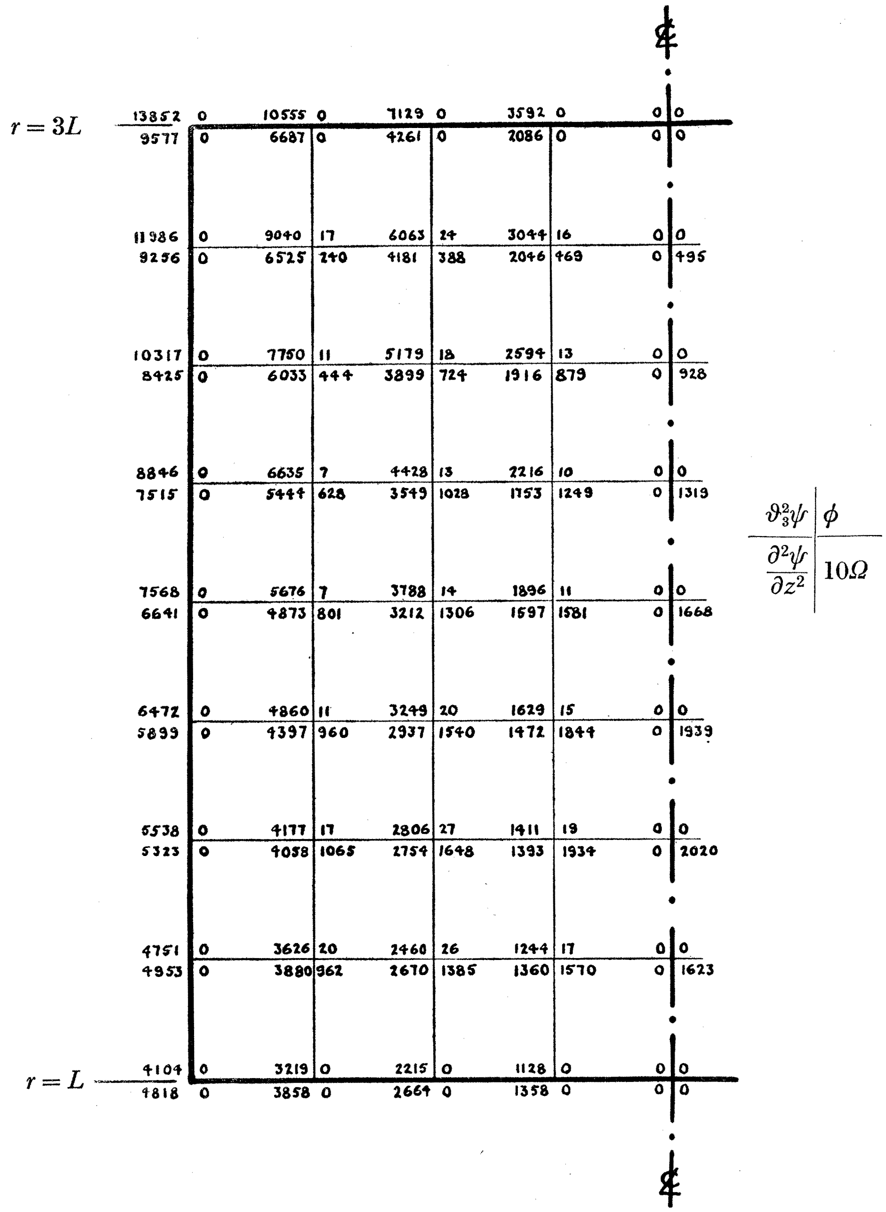

FIGURE 21

From this, as before, the six stress components were computed (by G. V.) in the manner of § 34. Figure 22 records the values of $\widehat{rr}$, $\widehat{\theta\theta}$, $\widehat{zz}$, $\widehat{zr}$, with contours exhibiting the variation of $\widehat{\theta\theta}$; in figure 23, values of $\widehat{r\theta}$ and $\widehat{\theta z}$ are recorded, and contours are drawn to exhibit the intensity of the resultant shear stress $\sqrt{(\widehat{r\theta}^2+\widehat{\theta z}^2)}$ at different points in the cross-section. Thus contours in figure 23 compare with the contours in figure 4, § 14: there, of course, only $\widehat{r\theta}$ and $\widehat{\theta z}$ had non-zero values.

39. Only two points call for special notice. First, on account of intrinsic errors in the finite-difference formulae, our approximation to $\oint\left\{r^3\frac{\partial\Omega}{\partial\nu}-\cos(r,\nu)\frac{\partial\chi}{\partial z}\right\}ds$ did not

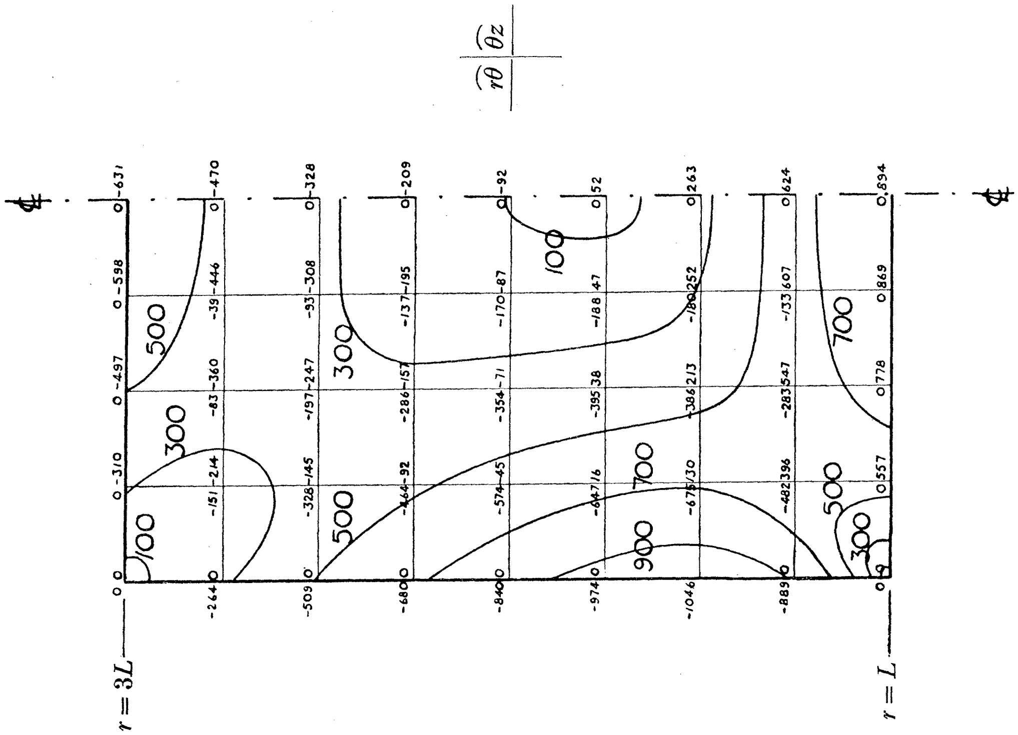

FIGURE 23. The curves are contours of resultant shear-intensity.

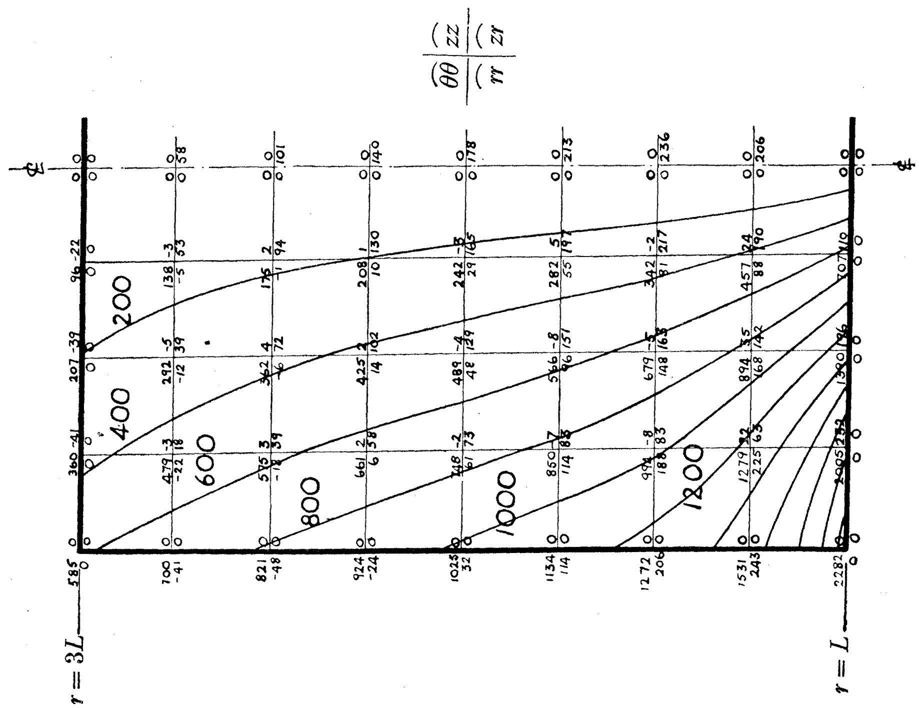

FIGURE 22. The curves are contours of $\widehat{\theta\theta}$.

vanish (except by reason of symmetry, in the 'Q-solution') in conformity with § 35, so our boundary values of $\partial^2\psi/\partial z^2$, deduced from (44), called for some correction to make them single-valued. Secondly, it was found that convergence (of the iterative process) could be accelerated by a timely use of 'optimal synthesis' (Part VII B, §§ 23–4) whereby two or more solutions were combined so as to reduce the sum total of the 'residual forces'. One such synthesis was used in deriving the 'Q-solution'.

Conclusion

40. In this paper Relaxation Methods have been shown to deal successfully, and to all the accuracy that will normally have value, with all of the problems covered (analytically) in a recent introductory paper (Southwell 1942). Those problems included all commonly occurring stress systems in solids of revolution, and perhaps have greater importance to the designer than the two-dimensional stress systems (plane stress or strain) which have received much more attention from elasticians—probably by reason of limitations inherent in orthodox methods of attack. Relaxation Methods appear to be the *only* practicable means of investigating stresses in solids of revolution, because (as the introductory paper emphasized) the cases which can be solved exactly and without excessive labour have, for the most part, only academic interest; whereas for problems in plane stress and strain, alternatives such as the photo-elastic technique are available.

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